IT'S ABOUT TIME:

Elementary Mathematical Reflections

On the Special and General

Theories of Relativity

Volume 3: *Mathematica* Notebooks and Solutions to the Problems

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Contents

Part 1

Mathematica Notebooks

The following notebooks are included in Volume 1. We are presenting them all together at this point for the reader's convenience. The reader who does not wish to key them in does not have to. They are available for download from my website at the University of Vermont:

Mathematica notebooks

MATHEMATICA NOTEBOOK 1

The Relativistic Law of Cosines

We begin by defining the function that gives the square of side v in terms of sides u and w and the cosine of angle ξ. (In this notebook, $x = Cos[\xi]$.)

```
f[u_, w_, x_] := (u^2 + w^2 - 2 u w x -
                          u^2 w^2 (1 - x^2)/c^2)/(1 - u w x/c^2)^2
```

Next, we subtract this computed value of the square of the side opposite ξ from the given value v^2 and eliminate all occurrences of w^2, replacing it with the value computed by applying this function to sides u and v and the function $\cos(\eta)$.

```
v^2 - f[u, w, x]/.w^2 -> f[u, v, y]
```

The value of this last output should be zero if x has the value it is asserted to have. We therefore replace x by the claimed value and see if we can coax *Mathematica* into telling us that the resulting value is zero. The "coaxing" requires some expansions and consolidation, and at one point it produces some terms containing w^2, which have to be eliminated again, but it does finally produce the output "0" that we were hoping for:

```
%/.x -> (u - v y)/(w (1 - u v y/c^2))
ExpandAll[%]
Together[%]
%/.w^2 -> f[u, v, y]
Together[%]
```

At this point, the output is 0, which means that the given expression for x is a root of the polynomial

```
p[x_] := %2  (1 - u w x/c^2)^2/.w^2 -> f[u, v, y]
```

which is the numerator of the expression for `v^2 - f[u,w,x]/.w^2 -> f[u,v,y]` given in the second input of the notebook (%2). Since `p[x]` is a quadratic polynomial in `x`, it has two roots, and we need to show that only the one just verified can be equal to $\cos\xi$. This is straightforward, since we can easily compute the coefficients of `p[t]`:

```
s = p[0]
r = p′[0]
q = p″[0]/2
s/q
```

This last expression is the product of the two roots of the polynomial, and the *Mathematica* output reveals that it is $c^4/(u^2w^2)$, which is larger than 1. Hence the

polynomial can have only one root in the interval $[-1, +1]$, namely the one we have already verified.

MATHEMATICA NOTEBOOK 2

The Velocity Tetrahedron

In this notebook, we use the variables $u1 = u'/c$, $v1 = v'/c$, $w1 = w'/c$, $t = \mathrm{Cos}\,[eta]$, $t1 = \mathrm{Cos}\,[\eta']$, $t2 = \mathrm{Cos}\,[\eta'']$, $a = u'v'\mathrm{Cos}\,[\omega]/c^2$. Here, x represents $(w/c)^2$. It is calculated from the law of cosines in the triangle $OO''O'''$ in the first statement, then each part of triangle $OO''O'''$ is replaced by its value as computed from the other three triangles in the second command. Finally, the output is simplified, and the result shows that x is also the value of $(w/c)^2$ computed from triangle XZY. Hence the expression given in the text really does represent w^2, as the equation asserts.

```
x = (u1^2 - 2 a + v1^2 - u1^2 v1^2 + a^2)/(1 - a)^2;
x /.  {u1^2 -> (u/c)^2 - 2 (u/c) w1 t1 + w1^2 -
(u/c)^2 w1^2 (1 - t1^2))/(1 - (u/c) w1 t1)^2,
v1^2 -> ((v/c)^2 - 2 (v/c) w1 t2 + w1^2 -
(v/c)^2 w1^2 (1 - t2^2))/(1 - (v/c) w1 t2)^2,
a -> ((w1 - u t1) (w1 - (v/c) t2) + (1 - w1^2) (u/c)(v/c)
(Cos[η] - t1 t2))/((1 - (u/c) w1 t1) (1 - (v/c) w1 t2))};
Together[%]
```

Associativity of Addition of Relativistic Velocities

The following sequence of commands illustrates the associativity of the "Lorentz addition" of triples defined in the text as $p(\theta, u, \varphi) \ +_L \ (\chi, v, \zeta) = (\theta+\xi, w, \psi)-\zeta)$. Here, we are concerned with three observers X, Y, and Z whose origins coincided at some epoch whose exact time coordinates are not relevant but may be assumed to be the instant that each of them takes as time 0. We are concerned with their mutual velocities, regarded as vectors in their common plane, at some later time, when any two of them are moving apart . One observer (X) sees a second observer (Y) moving at speed u in the direction making an angle θ with his positive x-axis, while Y sees X making angle φ with his positive x-axis. Similarly, Y sees a third observer (Z) moving at speed v in a direction making angle χ with that axis, while Z sees Y making angle ζ and X making angle ψ with his positive x-axis. Then X sees Z making angle $\theta+\xi$ with his positive x-axis and moving at speed w, while Z sees X making angle $\psi-\zeta$ with his positive x-axis . (All three speeds are positive numbers.) It is obvious geometrically and physically that this operation is associative . That means that if W is a fourth observer whose origin also coincided with the other three at time 0, and if W is now seen by Z to be moving at a speed t in a direction making angle τ with his x-axis, while W sees Z moving in a direction making angle σ with his x-axis, then we must have the equation $((\theta, u, \varphi) \ +_L \ (\chi, v, \zeta)) \ +_L \ (\tau, t, \sigma) = (\theta+\xi+v, s, \psi-\zeta-\sigma) = (\theta, u, \varphi) \ +_L \ ((\chi, v, \zeta) \ +_L \ (\tau, t, \sigma))$. While this equation is easy to prove using geometry, the algebra involved in simply grinding it out by brute computational force is too much even for *Mathematica*. Thus, instead of giving a very unenlightening computational proof, we prefer merely to illustrate the result by generating random values for the three addends and computing the difference between the two sides of this equation numerically, which should always be $\{0, 0, 0\}$. All angles are generated as random real numbers between 0 and 2π. All speeds are entered as random real multiples of c, the coefficient obviously being between 0 and 1. The binary operation is called "ladd" (Lorentz addition).

```
w[u_, v_, η_] := If[u == 0 c, v, If[v == 0 c, u,
If[Mod[η, 2 Pi] == Pi, (u + v)/(1 + u v/c^2),
If[Mod[η, 2 Pi] == 0, Abs[u - v]/(1 - u v/c^2),
FullSimplify[Sqrt[u^2 + v^2 - 2 u v Cos[η] -
u^2 v^2 Sin[η]^2/c^2]/(1 - u v Cos[η]/c^2)],
{c > u, c > v, u >= 0, v >= 0}]]]];
sgn[η_] := If[Mod[η, 2 Pi] == 0, 0, If[Mod[η, 2 Pi] == Pi, 0,
(Pi - Mod[η, 2 Pi])/Abs[Pi - Mod[η, 2 Pi]]]];
xinc[u_, v_, η_] := If[Mod[η, 2 Pi] == 0, 0,
If[Mod[η, 2 Pi] == Pi, (1 + sgn[η]) Pi/2,
ArcCos[(u - v Cos[η])/Sqrt[u^2 + v^2 - 2 u v Cos[η] -
```

```
u^2 v^2 Sin[η]^2/c^2]]]];
zinc[u_, v_, η_] := If[Mod[η, 2 Pi] == 0, 0,
If[Mod[η, 2 Pi] == Pi, (1 - sgn[η]) Pi/2,
ArcCos[(v - u Cos[ η])/Sqrt[u^2 + v^2 - 2 u v Cos[η] -
u^2 v^2 Sin[η]^2/c^2]]]];
ladd[{θ_, u_, φ_}, {ξ_, v_, ζ_}] := FullSimplify[{Mod[
sgn[u/c] (θ + sgn[ φ - ξ] xinc[u, v, φ - ξ]) + (1 - sgn[u/c]) ξ, 2 Pi],
w[u, v, φ - ξ],
Mod[sgn[v/c] (ζ - sgn[φ - ξ] zinc[u, v, φ - ξ]) +
(1 - sgn[v/c]) φ, 2 Pi]}, {c > u, c > v, u >= 0, v >= 0}];
a = Table[Table[{Random[Real, 2 Pi], Random[] c, Random[Real, 2 Pi]},
{k, 1, 3}], {j, 1, 5}];
Table[Chop[ladd[ladd[a[[j]][[1]], a[[j]][[2]]], a[[j]][[3]]] -
ladd[a[[j]][[1]], ladd[a[[j]][[2]], a[[j]][[3]]]]], {j, 1, 5}]
```

MATHEMATICA NOTEBOOK 4

Composition of Lorentz Transformations

The following notebook shows how to compose two Lorentz transformations, each of which is given in "privileged" coordinates, that is, the two pairs of observers X, Y and Y, Z whose space-time coordinates for an event are being converted both use their line of relative motion as the first axis of a coordinate system in space, the other two spatial axes being parallel at all times. Knowing the matrices of the two transformations in those special coordinates, we calculate the matrix of the conversion from X's coordinates to Z's in the privileged coordinates that those two observers share. The time coordinate is "spatialized" through multiplication by c, the speed of light.

The situation is as follows: We assume that X is sharing his first axis with Z. But we are not yet given the speed w of Z relative to X. Rather, our data consist of u (the speed of Y relative to X), v (the speed of Z relative to Y), and η, the angle between the directions that Y perceives between his lines of sight to X and Z, expressed as an angle (in radians) between 0 and π.

In order to communicate his coordinates to Z in the privileged coordinate system, X must first rotate the coordinates from the privileged coordinate system relative to Z to the privileged system relative to Y. That is done by multiplying those coordinates by the rotation matrix $x[u, v, \eta]$. Then the standard Lorentz transformation $l[u]$ adapted to the privileged coordinate system between X and Y converts them to Y' s coordinates in the privileged system relative to X. Next, Y rotates the coordinates through the angle $\pi - \eta$, thereby obtaining the coordinates in a system that is privileged relative to Z. After that, the standard Lorentz transformation $l[v]$ converts them to Z's privileged coordinate system relative to Y. Finally, Z rotates them, multiplying them by the matrix $z[u, v, \eta]$ so as to get them in Z's privileged system relative to X. The matrix product $z[u, v, \eta].l[v].r[\eta].l[u], x[u, v, \eta]$ is then the matrix of the Lorentz transformation between X and Z in the privileged coordinate systems they have relative to each other.

Examples show that the product does indeed have the simple form that it must have.

Two caveats accompany these instructions: (1) The velocities u and v must be entered as fractions (either finite-precision decimals or infinite-precision real numbers) of the speed of light c; (2) if the velocities are entered as infinite-precision irrational real numbers, *Mathematica* may display the same real number in two very different visual forms along the diagonal. Anyone who requires reassurance can have *Mathematica* do the algebra to verify that they are the same real number.

The notebook begins by determining w. Once that is done, theoretically, the two angles ξ and ζ can be determined from the law of cosines. Since the cosines of these angles appear in the rotation matrices that we need to define, however, we find it easier just to give two formulas for getting those angles out of the rotations, one

formula for the radian measure of the angles and another when the angle measure is wanted in degrees. Thus, in addition to its primary purpose of illustrating the way in which the composition of two Lorentz transformations is obtained, this program also provides a quick mechanical way of solving any relativistic velocity triangle given two of its sides and the included angle.

```
w[u_, v_, η_] := FullSimplify[Sqrt[u^2 + v^2 - 2 u v Cos[η]
            - u^2 v^2 Sin[η]^2/c^2]/(1 - u v Cos[η]/c^2),
                        {c > u, c > v, u > 0, v > 0, Pi > η, η > 0}];
α[u_] := FullSimplify[c/Sqrt[c^2 - u^2], {c > u, u > 0}];
l[u_] := {{α[u], - α[u] u/c, 0, 0}, {-α[u] u/c, α[u], 0, 0},
{0, 0, 1, 0}, {0, 0, 0, 1}};
r[η_] := {{1, 0, 0, 0}, {0, -Cos[η], -Sin[η], 0},
{0, Sin[η], -Cos[η], 0}, {0, 0, 0, 1}};
x[u_, v_, η_] := {{1, 0, 0, 0},
{0, (u - v Cos[η])/(w[u, v, η] (1 - u v Cos[η]/c^2)),
v Sin[η]/(α[u] w[u, v, η] (1 - u v Cos[η]/c^2)), 0},
{0, -v Sin[η]/(α[u] w[u, v, η] (1 - u v Cos[η]/c^2)),
(u - v Cos[η])/(w[u, v, η] (1 - u v Cos[η]/c^2)), 0}, {0, 0, 0, 1}};
xirad[u_, v_, η_] := ArcCos[x[u, v, η][[2]][[2]]];
xideg[u_, v_, η_] := ArcCos[x[u, v, η][[2]][[2]]]/Degree;
z[u_, v_, η_] := {{1, 0, 0, 0},
{0, (v - u Cos[η])/(w[u, v, η] (1 - u v Cos[η]/c^2)),
u Sin[η]/(α[v] w[u, v, η] (1 - u v Cos[η]/c^2)),0},
{0, - u Sin[η]/(α[v] w[u, v, η] (1 - u v Cos[η]/c^2)),
(v - u Cos[η])/(w[u, v, η] (1 - u v Cos[η]/c^2)), 0}, {0, 0, 0, 1}};
zetarad[u_, v_, η_] := ArcCos[z[u, v, η][[2]][[2]]];
zetadeg[u_, v_, η_] := ArcCos[z[u, v, η][[2]][[2]]]/Degree;
compose[u_, v_, η_] := Chop[FullSimplify[MatrixForm[
z[u, v, η].l[v].r[η].l[u].x[u, v, η]]]];
```

Standard Matrix of the Composition

We begin by defining the five matrices that we need to multiply: rotation through angle $-\xi$ (called *xi*), then the Lorentz transformation (called *uvel*) corresponding to velocity u (which converts X coordinates to Y coordinates with the principal axis for both X and Y along the common direction of motion), then rotation through angle θ (called *theta*), then the Lorentz transformation (called *vvel*) corresponding to velocity v (which converts Y coordinates to Z coordinates), and finally rotation through angle $-\zeta$ (called *zeta*) so as to get the Z coordinates in a system sharing its principal axis with X. We then multiply these five matrices to get the purported Lorentz transformation (whose matrix is called *lorcomp*) from X coordinates to Z coordinates.

```
xi = {{1, 0, 0, 0}, {0, Cos[ξ], -Sin[ξ], 0},
{0, Sin[ξ], Cos[ξ], 0}, {0, 0, 0, 1}};
uvel = {{α, -α u/c, 0, 0},
{-α u/c, α, 0, 0}, {0, 0, 1, 0},
{0, 0, 0, 1}};
theta = {{1, 0, 0, 0}, {0, Cos[θ], Sin[θ], 0}, {0, -Sin[θ],
Cos[θ], 0}, {0, 0, 0, 1}};
zeta = xi /.  ξ -> ζ;
vvel = uvel /.  {u -> v, α -> β};
lorcomp = zeta.vvel.theta.uvel.xi;
```

Nest, we insert the formulas that we know for the angles ξ and ζ in terms of u, v, and θ, renaming the modified matrix *lorcomp* as *lcm*:

```
lcm = lorcomp /.  {Cos[ζ] -> (v + u Cos[θ])/
Sqrt[(v + u Cos[θ])^2 + u^2 Sin[θ]^2/β^2],
Sin[ζ] -> u Sin[θ]/(β Sqrt[(v + u Cos[θ])^2 + u^2 Sin[θ]^2/β^2]),
Cos[ξ] -> (u + v Cos[θ])/ Sqrt[(u + v Cos[θ])^2 + v^2 Sin[θ]^2/α^2],
Sin[ξ] -> v Sin[θ]/(α Sqrt[(u + v Cos[θ])^2 + v^2 Sin[θ]^2/α^2])};
```

We now put in the given values of α and β in terms of u and v, again renaming the modified matrix, this time as *lcmp*:

```
lcmp = lcm /.  {α -> c/Sqrt[c^2 - u^2], β -> c/Sqrt[c^2 - v^2]};
```

Now we begin defining the entries in the matrix `lcmp` as new variables, so that we can simplify them one at a time:

```
lm11 = lcmp[[1]][[1]];
```

We want the matrix *lcmp* to be a standard Lorentz matrix like *uvel*, only with u replaced by w and α replaced by $\gamma = \alpha\beta\big(1 + \frac{uv\cos\theta}{c^2}\big)$. (The equality $\gamma =$

$c/\sqrt{c^2 - w^2} = \alpha\beta\big(1+\frac{uv\cos\theta}{c^2}\big)$ is easily verified by direct computation.) Accordingly, we make the following definitions:

```
w = Sqrt[(u + v Cos[θ])^2 +
v^2 Sin[θ]^2/α^2]/(1 +
u v Cos[θ]/c^2) /.  α ->
c/Sqrt[c^2 - u^2]; γ = c/Sqrt[c^2 - w^2];
γ^2 - (c^2/(c^2 - u^2)) (c^2/(c^2 - v^2)) (1 +
u v Cos[θ]/c^2)^2
Together[%]
FullSimplify[%]
```

We now verify that `lm11` $= \gamma$, as it should be. Since both $lm11$ and γ are positive, it suffices to show that their squares are equal. *Mathematica* can show this without much trouble:

```
γ^2 - lm11^2
Together[%]
FullSimplify[%]
```

We now systematically define $lmij$ as the entry in row i and column j of $lcmp$ and verify one by one that they are what they should be. Those in the fourth row or in the fourth column are immediately trivial. Some of the others require a little manipulation to make them work.

```
lm12 = lcmp[[1]][[2]];
lm12 + w lm11/c
Together[%]
lm13 = lcmp[[1]][[3]]
Together[%]
lm14 = lcmp[[1]][[4]]
lm21 = lcmp[[2]][[1]];
lm21 + w lm11/c
Together[%]
Numerator[%]
ExpandAll[%]
FullSimplify[%]
lm22 = lcmp[[2]][[2]];
lm22 - (c/Sqrt[c^2 - u^2])(c/Sqrt[c^2 - v^2])(1+u v Cos[θ]/c^2)
Together[%]
Numerator[%]
ExpandAll[%]
FullSimplify[%]
lm23 = lcmp[[2]][[3]]
Together[%]
Numerator[%]
FullSimplify[%]
```

```
lm24 = lcmp[[2]][[4]]
lm31 = lcmp[[3]][[1]]
Together[%]
lm32 = lcmp[[3]][[2]]
Together[%]
Numerator[%]
FullSimplify[%]
lm33 = lcmp[[3]][[3]]
FullSimplify[%]
lm34 = lcmp[[3]][[4]]
lm41 = lcmp[[4]][[1]]
lm42 = lcmp[[4]][[2]]
lm43 = lcmp[[4]][[3]]
lm44 = lcmp[[4]][[4]]
```

The Schwarzschild Metric

This notebook computes the parameters λ and ρ that occur in the space-time metric introduced by Schwarzschild. That is, it sets two independent components of the Ricci tensor equal to zero and solves the resulting differential equations. (The Ricci tensor has only two independent components in this case, and so there is no redundancy when the two are chosen.)

```
x = {t, ρ, φ, θ}; gsub = {{E^((λ[ρ])), 0, 0, 0},
{0, -E^(ν[ρ])/c^2, 0, 0}, {0, 0, -ρ^2/c^2, 0},
{0, 0, 0, -ρ^2 Sin[φ]^2/c^2}};
gsup = Inverse[gsub];
Γ = Table[(1/2) Sum[gsup[[i, l]] (D[gsub[[j, l]],x[[k]]] +
D[gsub[[l, k]], x[[j]]] - D[gsub[[j, k]], x[[l]]]), {l, 1, 4}],
{i,1,4}, {j,1,4}, {k,1,4}]
Ric = Table[Sum[(D[Γ[[i,j,l]],x[[i]]] - D[Γ[[i,j,i]], x[[l]]] +
Sum[Γ[[i,m,i]] Γ[[m,j,l]] - Γ[[i,l,m]] Γ[[m,i,j]], {m,1,4}]), {i,1,4}],
{j,1,4}, {l,1,4}]]
DSolve[{Ric[[1,1]] == 0, Ric[[2,2]] == 0}, {λ,ν}, ρ]
```

MATHEMATICA NOTEBOOK 7

Precession of Perihelion

This notebook computes the amount of relativistic precession of the perihelion of Mercury in each orbit of the sun, that is, it computes the difference between the difference in the polar angles (Mercury' s right ascension as seen from the sun) and one complete revolution (2π radians) that would be observed between successive perihelions if the solar system contained only Mercury and the sun. That difference is denoted $\Delta\theta - 2\pi$. It is then multiplied by 415 (the number of perihelions that occur in one century on earth) and converted to degrees to give the total amount of relativistic precession. (The actual observed precession is more than 100 times this amount, but the other 99+ percent of it can be accounted for as being due to perturbations of the other planets. In this notebook, we denote the Schwarzschild radius by A.

```
γ = e/(a(1 - e^2)); δ = 1/(a(1 - e^2));
rhs1[u_] := (2a(1 - e^2) - r0(3 + e^2))/(2a^2 (1 - e^2)^2) - u
+ (3r0/2)u^2;
rhs2[u_] := FullSimplify[2Integrate[rhs1[s], {s,1/(a(1 + e)), u}]];
rhs3[v_] := Factor[Together[ExpandAll[γ^(-2) rhs2[γv + δ]]]];
rhs4[φ_] = FullSimplify[Sec[φ]^2 rhs3[Sin[φ]]]];
```

At this point, if you ask it, *Mathematica* will tell you that rhs4 $[\varphi]$ equals

$$\frac{3r_s + a(-1+e^2) + r_s e\mathrm{Sin}\,[\varphi]}{a(-1+e^2)}\,.$$

From this point on, the reduction to a standard-form elliptic integral is the one given in the text. In order to compare theory with observation, we need numerical values for the parameters a, e, and r_s. The first two of these are obtained from the perihelion and aphelion distances ("rperi" and "raph") of Mercury from the sun. The third of them, by virtue of the inspired guess we made as to its value, depends on the universal gravitational constant G, the mass of the sun M, and the speed of light c. We ask *Mathematica* to store these as rational numbers, so that we can work with arbitrary precision later on (no matter how unrealistic it may seem to use 50 decimal places of an observed piece of data whose observational error may be only of the order of 10^{-5}). All distances are in meters, all masses in kilograms, and all velocities in meters per second.

```
c = 299792458; G = 667384/(10^(16)); M = 198892 x 10^(25);
raph = 698169 x 10^5; rperi = 460012 x 10^5;
a = (raph + rperi)/2; e = (raph - rperi)/(raph + rperi);
A = 2 G M/c^2;
```

MATHEMATICA NOTEBOOK 8

Curvature According to Euler

This notebook shows how to express the curvature of a surface in $\mathbb{R}^3$ that is parameterized as the graph of a function $h(x, y)$ in terms of the first- and second-order partial derivatives of $h(x, y)$. The curvature is computed using the method of Gauss and is then shown to be equivalent to the expression given by Euler.

```
r[x_, y_] := {x, y, h[x, y]};
n[x_, y_] := Cross[D[r[x, y], x], D[r[x, y], y]]/
Sqrt[Dot[Cross[D[r[x, y], x], D[r[x, y], y]],
Cross[D[r[x, y], x], D[r[x, y], y]]]];
e[x_, y_] := FullSimplify[Dot[D[r[x, y], x], D[r[x, y], x]]];
f[x_, y_] := FullSimplify[Dot[D[r[x, y], x], D[r[x, y], y]]];
g[x_, y_] := FullSimplify[Dot[D[r[x, y], y], D[r[x, y], y]]];
d[x_, y_] := FullSimplify[Dot[D[n[x, y], x], D[n[x, y], x]]];
d1[x_, y_] := FullSimplify[Dot[D[n[x, y], x], D[n[x, y], y]]];
d2[x_, y_] := FullSimplify[Dot[D[n[x, y], y], D[n[x, y], y]]];
sign[x_, y_] := FullSimplify[Sign[Dot[n[x, y],
Cross[D[n[x, y], x], D[n[x, y], y]]]],
1 + h^(0,1)[x,y]^2 + h^(1,0)[x,y]^2 > 0];
k[x_, y_] := sign[x, y] FullSimplify[Sqrt[(d[x, y] d2[x, y]
- d1[x, y]^2)/(e[x, y] g[x, y] - f[x, y]^2)],
{1 + h^(0,1)[x,y]^2 + h^(1,0)[x,y]^2 >0,
(h^(1,1)[x,y]^2 - h^(0,2)[x,y] h^(2,0)[x,y])^2 >0}]; k[x, y]
```

MATHEMATICA NOTEBOOK 9

Curvature of Surfaces in $\mathbb{R}^3$

In this notebook, we give an algorithm for computing the curvature of a parameterized surface embedded in three-dimensional Euclidean space, one that involves only the metric coefficients of the surface and their first and second derivatives.

We begin by defining the matrix `gsub`, whose entries are the metric coefficients, that is, the coefficients of the first fundamental form.

Denoting the inverse of `gsub` by `gsup`, we define the eight Christoffel symbols in the usual way as combinations of the metric coefficients and their first-order partial derivatives.

```
x = {u, v}; gsub[u_, v_] := {{e[u, v],f[u, v]}, {f[u, v], g[u, v]}};
gsup[u_, v_] := Inverse[gsub[u, v]];
Γ[u_, v_] := FullSimplify[Table[(1/2) Sum[gsup[u, v][[i,l]]
(D[gsub[u, v][[j, l]], x[[k]]] + D[gsub[u, v][[l, k]], x[[j]]]
- D[gsub[u, v][[j, k]], x[[l]]]), {l, 1, 2}], {i, 1, 2}, {j, 1, 2},
{k, 1, 2}]];
```

We now define the curvature in terms of the Christoffel symbols and their first-order partial derivatives, together with the metric coefficients.

```
κ[u_, v_] := FullSimplify[Table[Sum[gsub[u, v][[m, i]]
(D[Γ[u, v][[m, l, j]], x[[k]]] - D[Γ[u, v][[m, k, j]], x[[l]]]
+ Sum[Γ[u, v][[n, l, j]] Γ[u, v][[m, k, n]]
- Γ[u, v][[n, k, j]] Γ[u, v][[m,l, n]], {n, 1, 2}]), {m, 1, 2}]/
(e[u, v] g[u, v] - f[u, v]^2), {l, 1, 2},
{k, 1, 2}, {j, 1, 2}, {i, 1, 2}]];
```

Finally, we feed our formula some metric coefficients as test cases.

1. The cubic surface $z = x^3 - y^2$, taking $u = x$ and $v = y$ as parameters.

```
e[u_, v_] := 1 + 9 u^4; f[u_, v_] := -6 u^2 v;
g[u_, v_] := 1 + 4 v^2; κ[u, v]
```

2. The hyperbolic plane in polar coordinates, for which the metric coefficients are worked out in the text. We use r_0 rather than k for the radius of curvature, since k is a dummy variable of summation in our definition.

```
e[u_, v_] := r_0^2/(u^2); f[u_, v_] := 0; g[u_, v_] := u^2; κ[u, v]
```

3. The pseudosphere, which is an isometric representation of the hyperbolic plane:

```
e[u_, v_] := 1; f[u_, v_] := 0; g[u_, v_] := r_0^2 Sinh[u/r_0]^2;
κ[u, v]
```

4. The torus:

```
e[u_, v_] := r^2; f[u_, v_] := 0; g[u_, v_] := (r_0 + r Cos[u])^2;
κ[u, v]
```

5. The hyperbolic paraboloid whose equation is $z = (x^2 - y^2)/a$.

```
e[u_, v_] := 1 + 4 u^2/a^2; f[u_, v_] := -4 u v/a^2;
g[u_, v_] := 1 + 4 v^2/a^2; κ[u, v]
```

6. The disk of radius c, whose points are regarded as relativistic velocities:

```
e[u_, v_] := 1/(1 - u^2/c^2)^2; f[u_, v_] := 0;
g[u_, v_] := u^2/(1 - u^2/c^2); κ[u, v]
```

7. The rotating plane of "relativistic" Ptolemaic astronomy:

```
e[u_, v_] := 1 + (2 Pi u/(c T))^2 ((2 Pi u/(c T))^4
- 3 (2 Pi u/(c T))^2 + 3)/(1 - (2 Pi u/(c T))^2)^3;
f[u_, v_] := 0; g[u_, v_] := u^2; κ[u, v]
```

8. A generic surface of revolution with the equation $z = h(r)$, the radial coordinate r playing the role of the parameter u.

```
e[u_, v_] := 1 + h′[u]^2; f[u_, v_] := 0; g[u_, v_] := u^2; κ[u, v]
```

MATHEMATICA NOTEBOOK 10

Sectional Curvature of $\mathbb{S}^3$

The following program uses the Riemann curvature tensor $R(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w}$ to compute the sectional curvature of the three-dimensional sphere $\mathbb{S}^3$ of radius r_0 in $\mathbb{R}^4$. For any two linearly independent tangent vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ this curvature, denoted κ, is equal to $\langle \boldsymbol{u}, R(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{v}\rangle / (\langle \boldsymbol{u}, \boldsymbol{u}\rangle \langle \boldsymbol{v}, \boldsymbol{v}\rangle - \langle \boldsymbol{u}, \boldsymbol{v}\rangle^2)$

We begin by parameterizing this manifold using one longitude angle θ and two latitude angles φ and ψ. We then compute the metric coefficients as the 3×3 matrix gsub whose entries are g_{ij} and its inverse gsup, whose entries are g^{ij}. Then we compute the 27 Christoffel symbols Γ^i_{kj} and define the inner product $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = g_{ij} u^i v^j$ on the tangent space T as the function anglebracket[$\boldsymbol{u}, \boldsymbol{v}$]. None of this output needs to be displayed, and so we suppress it.

```
r[θ_, φ_, ψ_] := {r0 Cos[θ] Cos[φ] Cos[ψ],
r0 Sin[θ] Cos[ φ] Cos[ψ], r0 Sin[φ] Cos[ψ], r0 Sin[ψ]};
x = {θ, φ, ψ};
gsub = Table[FullSimplify[Dot[D[r[θ, φ, ψ], x[[i]]],
D[r[θ, φ, ψ], x[[j]]]]],{j, 1, 3}, {i, 1, 3}];
gsup = Inverse[gsub];
Γ = FullSimplify[Table[(1/2) Sum[gsup[[i,q]](D[gsub[[j,q]], x[[k]]] +
D[gsub[[q,k]], x[[j]]] - D[gsub[[j,k]], x[[q]]]),
{q, 1, 3}], {i, 1, 3}, {j, 1, 3}, {k, 1, 3}]];
dot[y_, z_] := Sum[gsub[[i, j]] y[[i]] z[[j]], {i, 1, 3}, {j, 1, 3}];
```

Next, we compute and display the 81 coordinates of the Riemann curvature tensor.

```
Riem = FullSimplify[Table[D[Γ[[i,j,l]], x[[k]]]
- D[Γ[[i,j,k]], x[[l]]] +
Sum[Γ[[n,j,l]] Γ[[i,k,n]] - Γ[[n,j,k]] Γ[[i, l, n]],{n, 1, 3}],
{i, 1, 3}, {j, 1, 3}, {k, 1, 3}, {l, 1, 3}]];
```

We are now prepared to define the Riemann curvature tensor as a mapping from $T \times T \times T$ into T.

```
R[u_, v_, w_] := Table[Sum[Riem[[i, j, k, l]] u[[k]] v[[l]] w[[j]],
{j, 1, 3}, {k, 1, 3}, {l, 1, 3}], {i, 1, 3}]
```

Finally, we take a random pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and compute the curvature κ. This curvature is actually independent of the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, provided they are linearly independent.

```
u = {1, 3, 2}; v = {4, -1, 5}; κ = FullSimplify[
dot[u,R[u, v, v]]/(dot[u, u] dot[v, v] - dot[u, v]^2)]
```

MATHEMATICA NOTEBOOK 11

The Contravariant Einstein Tensor

We first define labels for the coordinates and general metric coefficients, the non-diagonal coefficients being set defined as 0. After that, we use the standard computational definitions for the inverse of the matrix of metric coefficients, the tableau (Γ) of Christoffel symbols, the Riemann curvature tensor Riem, the Ricci tensor Ric, the scalar curvature R, the covariant Einstein tensor Ein, and finally the contravariant Einstein tensor ContraEin. There is no need to see any of this output, but the reader who is curious and wishes to take a look will be suitably impressed by the complexity of all of them.

```
p = {t, x, y, z}; gsub = {{g11[t, x, y, z], 0, 0, 0},
{0, g22[t, x, y, z] , 0, 0}, {0, 0, g33[t, x, y, z], 0},
{0, 0, 0, g44[t, x, y, z]}}; gsup = FullSimplify[Inverse[gsub]];
Γ = FullSimplify[Table[(1/2) Sum[gsup[[i, l]]
(D[gsub[[j, l]], p[[k]]] + D[gsub[[l, k]], p[[j]]]
- D[gsub[[j, k]], p[[l]]]), {l, 1, Length[p]}],
{i, 1, Length[p]}, {j, 1, Length[p]}, {k, 1, Length[p]}]];
Riem = FullSimplify[Table[D[Γ[[i, l, j]], p[[k]]]
- D[Γ[[i, k, j]], p[[l]]] + Sum[Γ[[n, l, j]] Γ[[i, k, n]]
- Γ[[n, k, j]] Γ[[i, l, n]], {n, 1, Length[p]}],
{i, 1, Length[p]}, {j, 1, Length[p]}, {k, 1, Length[p]},
{l, 1, Length[p]}]]; Ric = FullSimplify[Table[Sum[Riem[[i, j, i, l]],
{i, 1, Length[p]}], {j, 1, Length[p]}, {l, 1, Length[p]}]];
R = FullSimplify[Sum[gsup[[i, j]] Ric[[i, j]], {i, 1, Length[p]},
{j, 1, Length[p]}]]; Ein = FullSimplify[Ric - (R/2) gsub];
ContraEin = FullSimplify[Table[Sum[gsup[[i, k]] gsup[[j, m]]
Ein[[k, m]], {k, 1, Length[p]}, {m, 1, Length[p]}], {i, 1, Length[p]},
{j, 1, Length[p]}]];
```

We then give the instructions for computing the divergence of the contravariant tensor ContraEin. This time, we do wish to see the output, hoping it will be $\{0, 0, 0, 0\}$. (Spoiler alert: It will be!)

```
div = FullSimplify[Table[Sum[D[ContraEin[[i, j]], p[[j]]] +
Sum[ContraEin[[j, q]] Γ[[i, q, j]] + ContraEin[[i, q]] Γ[[j, q, j]],
{q, 1, Length[p]}], {j, 1, Length[p]}], {i, 1, Length[p]}]]
```

MATHEMATICA NOTEBOOK 12

Geodesics on the Pseudo-sphere

This notebook makes it possible to plot geodesics on the pseudo-sphere, provided the project orthogonally into the first quadrant of the circle of radius k in the xy-plane.

```
k = 10; r1 = 4k/5; θ1 = Pi/6; r2 = 3k/5; θ2 = 0; ρ[r_, θ_] := {r Cos[θ],
r Sin[θ],
k Log[(k + Sqrt[k^2 - r^2])/r] - Sqrt[k^2 - r^2]};
w[u_, v_, t_] := k u/Sqrt[k^2 - u^2 (t - v)^2];
θ3 = y /.  NSolve[{r1^2 (k^2 - x^2 (y - θ1)^2) ==
k^2 x^2, r2^2 (k^2 - x^2 (y - θ2)^2) == k^2 x^2}, {x, y}][[1]];
r3 = x /.  NSolve[{r1^2 (k^2 - x^2 (y - θ1)^2) ==
k^2 x^2, r2^2 (k^2 - x^2 (y - θ2)^2) == k^2 x^2}, {x, y}][[1]];
a = ParametricPlot3D[{r Cos[θ], r Sin[θ],
k Log[(k + Sqrt[k^2 - r^2])/r] - Sqrt[k^2 - r^2]},
{r, 1, k}, {θ, 0, 2 Pi},
AspectRatio -> Automatic, Axes -> False, Boxed -> False];
b = ParametricPlot3D[ρ[w[r3, θ3, t], t], {t, θ1,θ2},
AspectRatio -> Automatic, Axes -> False, Boxed -> False];
Show[a, b, PlotRange -> All]
```

Part 2

Solutions to the Problems

CHAPTER 1

Time, Space, and Space-time

PROBLEM 1.1. Solve the equations of the Lorentz transformation for t, x, y, and z in terms of t', x', y', and z', and show that the solution is the same transformation with u replaced by $-u$ (which makes no change in α).

Solution: It is trivial that $y = y'$ and $z = z'$. (Distances perpendicular to the line of relative motion are preserved.) It is the mixing of time with distance along the line of relative motion that makes relativistic kinematics different from the Newtonian variety.

Eliminating tr from the first two equations results in the equation $y + ut' = \alpha(1 - u^2/c^2)x' = x/\alpha$, and hence $x = \alpha(ut' + x')$, that is, it is obtained from the equation for x' by interchanging x and x' and t and $t's$ and replacing u with $-u$.

Similarly, eliminating x from these two equations yields the equation $(t' + ux'/c^2) = \alpha t(1 - u^2/c^2) = t/\alpha$, and hence $t = \alpha(t' + ux'/c^2)$, which is the first of the equations, with t and t' interchanged, x and x' interchanged, and u replaced with $-u$.

PROBLEM 1.2. Revisit the problem of the twin paradox by imagining that Mary has a telescope trained on the earth, so that she can constantly observe John's clock. What would she see? Why is it that this clock shows a later time than Mary's own clock when the twins meet at the end of the journey?

Solution: Supposing that Mary measures her proper time t from the instant when she passed her brother on her outward journey, light from the earth reaching her at time t will have set out at a time (on Mary's clock) $s < t$ when earth was (according to Mary) located at the point $-su$ on her axis. Thus, since the light has been traveling for a time $t - s$, we have

$$c(t - s) = su\,,$$

which means that

$$s = \frac{ct}{c+u}\,.$$

Thus, making proper Newtonian allowance for the time light required to reach her, Mary says that the time shown on John's clock is an event that took place at location $-su$ and at time s. By the Lorentz transformation, John records this event as having occurred at $\alpha(-su + us) = 0$ (naturally) and at time $s' = \alpha(s - su^2/c^2) = s/\alpha$. This time, which is what Mary will observe on the face of John's clock when she looks through her telescope, is indeed less than the time Mary calculates in her own proper time as the time of that event. In fact

$$s' = \frac{ct}{\alpha(c+u)} = \frac{\sqrt{c^2 - u^2}t}{c+u} = \sqrt{\frac{c-u}{c+u}}t\,.$$

We recognize the relativistic Doppler shift in this formula. When Mary arrives at her destination, at time $t_1 = d/(\alpha u)$ on her clock, the time s_1 that she sees on John's clock is given by

$$s_1 = \Big(\frac{1}{u} - \frac{1}{c}\Big)d\,.$$

Note that $s_1 + d/c = d/u$ is the time John would calculate as the time when this signal (picture of the clock face, if you like) will arrive at the star, as it should be, since it is arriving there simultaneously with Mary.

To see what happens on the return journey, all we have to do is replace u by $-u$. Now the relativistic Doppler shift goes the other way. That is, for times $t > t_1$ measured from the time of Mary's departure for earth, we have

$$s - s_1 = \sqrt{\frac{c+u}{c-u}}(t - t_1)\,,$$

and when Mary arrives back at the earth, at time $t_2 = 2t_1$, the time shown on John's clock will be

$$s_2 = s_1 + \sqrt{\frac{c+u}{c-u}}t_1 = \Big(\sqrt{\frac{c-u}{c+u}} + \sqrt{\frac{c+u}{c-u}}\Big)t_1 = \alpha(2t_1) = \alpha t_2\,,$$

which is indeed larger than the time t_2 on Mary's clock. An equivalent statement is that

$$s_2 - s_1 = \Big(\frac{1}{u} + \frac{1}{c}\Big)d\,,$$

from which we infer, taking account of the value of s_1, that $s_2 = 2d/u$.

There is an analogy here with the well-known description of a person rowing in a river. The person is presumed to be able to row at speed u faster than the current v. A journey upriver a distance d, then back to the starting point, requires time

$$\frac{d}{u-v} + \frac{d}{u+v} = \frac{2u}{u^2 - v^2}d\,.$$

But on a quiet lake, the same rower can go the same distance in time $2d/u$, which is a shorter time. The effect we are seeing here resembles the Doppler shift, as the medium in which the boat is moving has a current that accelerates the rower in one direction and retards him in the other. The two effects do not balance out over a constant *distance* upriver and back, since the increased velocity in one direction is not maintained as long as the decreased velocity in the other direction. The rower's average speed, which is u in quiet water, is $u(1 - v^2/u^2)$ in this upriver-downriver journey. If the upstream and downstream legs of the journey lasted the same amount of *time*, the average speed would be u, but of course, the rower would finish downstream of the starting point.

There is a significant difference between the rower and the astronaut, however. In the interstellar journey, the "rower" is the time on the earth-based clock seen through the telescope by the astronaut. In this situation, the increased speed (relative to the speed of the hands on the astronaut's clock) is maintained exactly as long as the decreased speed. As just stated, in the classical case, that would leave the average speed unchanged. But, because the coefficients of increase and decrease are those of the *relativistic* Doppler shift rather than the classical ones, the increase is greater than the decrease, that is,

$$\sqrt{\frac{c+u}{c-u}} - 1 > 1 - \sqrt{\frac{c-u}{c+u}}\,.$$

As a result, at the end of the journey, the hands on the earth-based clock will be ahead of those on the traveling clock.

PROBLEM 1.3. Consider the vector formulation of the Lorentz transformation given by the mutually inverse relations

$$(t'; \boldsymbol{x}') = \left(\alpha\Big(t - \frac{\boldsymbol{u}\cdot\boldsymbol{x}}{c^2}\Big); \boldsymbol{x} + \Big((\alpha-1)\frac{\boldsymbol{x}\cdot\boldsymbol{u}}{\boldsymbol{u}\cdot\boldsymbol{u}} - \alpha t\Big)\boldsymbol{u}\right)$$

and

$$(t, \boldsymbol{x}) = \left(\alpha\Big(t' + \frac{\boldsymbol{u}\cdot\boldsymbol{x}'}{c^2}\Big); \boldsymbol{x}' + \Big((\alpha-1)\frac{\boldsymbol{x}'\cdot\boldsymbol{u}}{\boldsymbol{u}\cdot\boldsymbol{u}} + \alpha t'\Big)\boldsymbol{u}\right).$$

Verify that these relations really are inverses of each other by inserting the values of $\boldsymbol{x}$ and t from the second relation into the right-hand side of the first relation.

Solution: This problem is a simple matter of gathering like terms. Assuming the second equation holds, its vector part yields the relation $\boldsymbol{x}\cdot\boldsymbol{u} = \alpha(\boldsymbol{x}'\cdot\boldsymbol{u} + u^2t')$. The vector portion of the right-hand side of the first equation then becomes

$$t'\boldsymbol{x}' + \Big((\alpha-1)\frac{\boldsymbol{x}'\cdot\boldsymbol{u}}{u^2} + \alpha t'\Big)\boldsymbol{u} + \left(\alpha(\alpha-1)\Big(\frac{\boldsymbol{y}\cdot\boldsymbol{u}}{u^2} + t'\Big) - \alpha^2 t' - \alpha^2\frac{\boldsymbol{x}'\cdot\boldsymbol{u}}{c^2}\right)\boldsymbol{u}\,,$$

and this expression collapses to

$$\boldsymbol{x}' + \left(\Big(\frac{\alpha-1}{u^2} + \frac{\alpha(\alpha-1)}{u^2} - \frac{\alpha^2}{c^2}\right)\boldsymbol{u}\,.$$

Here, however, the coefficient of $\boldsymbol{u}$ is equal to

$$\frac{(\alpha-1)c^2 + \alpha(\alpha-1)c^2 - \alpha^2u^2}{u^2c^2}\,,$$

which in turn is simply

$$\frac{\alpha^2(c^2-u^2) - c^2}{u^2c^2} = 0\,.$$

PROBLEM 1.4. Show that the observers O and O' agree about the "space-time metric," that is, show that $(ct^2) - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2$.

Solution: This is a routine computation, using the equations of the Lorentz transformation:

$$\begin{aligned}
(ct')^2 - x'^2 - y'^2 - z'^2 &= \big(\alpha(ct - ux/c)\big)^2 - \big(\alpha(-ut + x)\big)^2 - y^2 - z^2\\
&= \alpha^2\big(c^2t^2 - 2tux + u^2x^2/c^2\big)\\
&\qquad - \alpha^2(u^2t^2 - 2utx + x^2) - y^2 - z^2\\
&= \alpha^2(c^2-u^2)t^2 + \alpha^2(u^2/c^2 - 1)x^2 - y^2 - z^2\\
&= (ct)^2 - x^2 - y^2 - z^2\,.
\end{aligned}$$

PROBLEM 1.5. Let a and b be dimensionless positive constants. Show that the mutually perpendicular lines $bx = ay$, $z = 0$ and $by = -ax$, $z = 0$ observed by O make the non-obtuse angle $\arccos\big(ab(\alpha^2-1)/\sqrt{(a^2\alpha^2+b^2)(b^2\alpha^2+a^2)}\big)$ when observed by O' at any given instant t'. Show that this is a right angle only if $u = 0$, and that it tends to $0°$ as $u \uparrow c$.

Solution: The equation $bx = ay$ translates to $b\alpha(ut'+x') = ay'$, that is, $b\alpha x' - ay' = -b\alpha ut'$. Similarly, the line $by = -ax$ has equation $by' = -a\alpha(ut' + x')$, so that $a\alpha x' + by' = -a\alpha ut'$. The plane vectors perpendicular to these two lines are $(a, b\alpha)$ and $(-b, a\alpha)$. The cosine of the angle between them is

$$\frac{(\alpha^2 - 1)ab}{\sqrt{a^2\alpha^2 + b^2}\sqrt{b^2\alpha^2 + a^2}}\,.$$

Given that a and b are neither one equal to zero, this cosine can vanish only when $\alpha = 1$, which occurs only when $u = 0$. As $u \uparrow c$, α tends to infinity, and when we divide the numerator and denominator of this fraction by α^2, then let α tend to infinity, the limit of the fraction is 1, that is, the angle is 0.

Problem 1.6. Translate the equation of the unit circle $x^2+y^2 = R^2$, as seen by O, into O''s coordinate system. What kind of curve does this equation represent? How does the shape depend on time?

Solution: From the Lorentz transformation we get

$$\alpha^2(ut' + x')^2 + (y')^2 = R^2\,.$$

This is the equation of an ellipse. In standard form

$$\frac{(x' - h)^2}{a^2} + \frac{(y')^2}{b^2} = 1\,,$$

where

$$h = -ut'\,,\ \ a = \frac{R}{\alpha}\,,\ \ b = R\,.$$

As these equations show, the ellipse is (of course) traveling with speed $-u$ along the x'-axis. Its major axis is the y'-axis, and its minor axis is parallel to the x'-axis.

The shape of the ellipse, that is, the ratio of its major axis to its minor axis, is constant, namely α. Its eccentricity is $1-1/\alpha^2 = u^2/c^2$. Neither of these quantities depends on time, only on the relative velocity of the observers. As $u \uparrow c$, the ellipse approaches a parabola.

Problem 1.7. Translate the equation of a general conic section $Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$, observed by O, into the coordinate system used by O', getting an equation $A'x'^2 + 2B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$. Show that the discriminant $\Delta = B^2 - AC$ becomes $\Delta' = (B')^2 - A'C' = \alpha^2(B^2 - AC) = \alpha^2\Delta$. In particular, an ellipse ($\Delta < 0$) remains an ellipse (although, as Problem 1.6 shows, a circle may become a more general ellipse), a parabola ($\Delta = 0$) remains a parabola, and a hyperbola ($\Delta > 0$) remains a hyperbola.

Solution: Using the Lorentz equations, we get the equation

$$A\alpha^2(-ut' + x')^2 + 2B\alpha(-ut' + x')y' + Cy'^2 + D\alpha(-ut' + x') + Ey' + F = 0\,,$$

which, when expanded and rearranged, yields

$$\begin{aligned}
A' &= A\alpha^2\\
B' &= B\alpha\\
C' &= C\\
D' &= D\alpha - 2Aut'\alpha^2\\
E' &= E - 2But'\alpha^2\\
F' &= F - D\alpha ut' + A\alpha^2u^2(t')^2\,.
\end{aligned}$$

We then get $(B')^2 - A'C' = \alpha^2 B^2 - \alpha^2 A^2 C^2 = \alpha^2(B^2 - AC)$, as asserted.

REMARK 1.1. This result is not at all peculiar to the Lorentz transformation: *Any* invertible linear operator T on $\mathbb{R}^2$ will preserve the three types of conic sections. The general relationship is $\Delta' = \big(\det(T)\big)^2\Delta$.

PROBLEM 1.8. Consider the mapping $f(u)$ from $(-c, c)$ onto $(-\infty, \infty)$ given by

$$x = f(u) = \log\left(\frac{c+u}{c-u}\right).$$

(The base of the logarithm is not important here. It may be any positive number except 1.)

Find the inverse of the mapping $f(u)$. Also prove that for the Lorentz addition of velocities $u * v = (u+v)/(1+uv/c^2)$,

$$f(u*v) = f(u) + f(v).$$

Thus, the group of relativistic velocities in one dimension is isomorphic to the additive group of real numbers. (That is no surprise, since, up to isomorphism, this is the only non-compact connected one-dimensional real Lie group that exists.)

Solution: Since $e^x = (c+u)/(c-u)$, simple algebra reveals that

$$u = c\Big(\frac{e^x - 1}{e^x + 1}\Big) = c\Big(\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}\Big) = c\tanh\left(\frac{x}{2}\right).$$

We also have

$$\begin{aligned}
f(u*v) &= \log\left(\frac{c+u*v}{c-u*v}\right)\\
&= \log\left(\frac{c+u+v+\frac{uv}{c}}{c-u-v+\frac{uv}{c}}\right)\\
&= \log\left(\frac{c^2+cu+cv+uv}{c^2-cu-cv+uv}\right)\\
&= \log\left(\frac{(c+u)(c+v)}{(c-u)(c-v)}\right)\\
&= \log\left(\frac{c+u}{c-u}\right) + \log\left(\frac{c+v}{c-v}\right)\\
&= f(u) + f(v).
\end{aligned}$$

PROBLEM 1.9. Suppose that O' observes O and O'' moving away at constant speeds u and v along lines making angle η. Let w be the speed with which O and O'' are moving apart relative to each other, and let

$$\alpha = \frac{c}{\sqrt{c^2-u^2}}\,;\quad \beta = \frac{c}{\sqrt{c^2-v^2}}\,;\quad \gamma = \frac{c}{\sqrt{c^2-w^2}}\,.$$

Show that

$$\gamma = \alpha\beta\Big(1 - \frac{uv\cos\eta}{c^2}\Big).$$

Thus, the angles of a relativistic velocity triangle are determined by the its sides u, v, and w via the formula

$$\cos\eta = \frac{c^2}{uv}\left(1 - \frac{\gamma}{\alpha\beta}\right),$$

for the angle opposite side of length w and the analogous formulas for the other two angles.

Solution: This formula is a routine computation, starting from the basic formula for w, written in terms of u, v, and $\cos\eta$. We have

$$w^2 = \frac{u^2 + v^2 - 2uv\cos\eta - u^2v^2\sin^2\eta/c^2}{\left(1 - uv\cos\eta/c^2\right)^2}\,.$$

It follows easily that

$$(c^2-w^2)(1-uv\cos\eta/c^2)^2 = c^2(1-uv\cos\eta/c^2)^2 - \left(u^2+v^2-2uv\cos\eta-u^2v^2\sin^2\eta/c^2\right).$$

The right-hand side now collapses to $(c^2 - u^2)(c^2 - v^2)/c^2$, yielding

$$\frac{c^4}{(c^2-u^2)(c^2-v^2)}\left(1 - \frac{uv\cos\eta}{c^2}\right)^2 = \frac{c^2}{c^2-w^2}\,.$$

The formula for γ is obtained simply by taking the square root on both sides. The resulting equation is then easily solved for $\cos\eta$, yielding the expression given above.

Problem 1.10. Show that if ξ and η and v and w are interchanged, then Eqs. (1), (2), and (3) below remain valid. Likewise, if ζ and η and u and w are interchanged, then Eqs. (1), (4), and (5) remain valid. It follows from Eqs. (3) and (5) that

$$\frac{\beta v}{\alpha u} = \frac{v\sqrt{1-u^2/c^2}}{u\sqrt{1-v^2/c^2}} = \frac{\sin\xi}{\sin\zeta}\,.$$

This relation is the law of sines for a relativistic velocity triangle. Formally, it is the ordinary law of sines applied to a triangle in which the sides opposite the two angles are shrunk, each by the FitzGerald–Lorentz contraction factor that would be measured by the observer corresponding to the opposite vertex.

Solution: For ease of reference, the equations in question are

$$\begin{aligned}
(1)\ w^2 &= \frac{u^2+v^2-2uv\cos\eta-u^2v^2(\sin^2\eta)/c^2}{\left(1-uv(\cos\eta)/c^2\right)^2}\,.\\
(2)\ \cos\xi &= \frac{u-v\cos\eta}{\sqrt{u^2+v^2-2uv\cos\eta-u^2v^2(\sin^2\eta)/c^2}} = \frac{u-v\cos\eta}{w(1-uv(\cos\eta)/c^2)}\,,\\
(3)\ \sin\xi &= \frac{v\sin\eta}{\alpha\sqrt{u^2+v^2-2uv\cos\eta-u^2v^2(\sin^2\eta)/c^2}} = \frac{v\sin\eta}{\alpha w(1-uv(\cos\eta)/c^2)}\,,\\
(4)\ \cos\zeta &= \frac{v-u\cos\eta}{\sqrt{u^2+v^2-2uv\cos\eta-u^2v^2(\sin^2\eta)/c^2}} = \frac{v-u\cos\eta}{w(1-uv(\cos\eta)/c^2)}\,,\\
(5)\ \sin\zeta &= \frac{u\sin\eta}{\beta\sqrt{u^2+v^2-2uv\cos\eta-u^2v^2(\sin^2\eta)/c^2}} = \frac{u\sin\eta}{\beta w(1-uv(\cos\eta)/c^2)}\,,
\end{aligned}$$

where, of course, $\alpha = c/\sqrt{c^2-u^2}$ and $\beta = c/\sqrt{c^2-v^2}$.

To show that Eq. 1 remains valid under the stated interchanges, all one has to do is note the relative positions of the parts in question.

Problem 1.11. If $\eta = 0$ and $u = v$, then $w = |\boldsymbol{u} - \boldsymbol{v}| = 0$. The expression given by Eq. (1) in the previous problem for w shows that it is very unlikely that w^2 can ever equal $|\boldsymbol{u}-\boldsymbol{v}|^2 = u^2 - 2uv\cos\eta + v^2$ in any other case. Because of

the denominator $\bigl(1 - uv\cos\eta/c^2\bigr)^2$, this certainly cannot happen unless η is an acute angle. Show that this can nevertheless occur at any speed in the case of an "isosceles" triangle corresponding to $u = v$, given a suitable vertex angle. To do so, assume $u = v$ and show that the equation $w = |\boldsymbol{u} - \boldsymbol{v}|$ leads to the quadratic equation $2a^2x^2 - 3x + 1 = 0$ for the unknown $x = \cos\eta$, where a $(0 \le a < 1)$ is a dimensionless constant, namely $a = u/c = v/c$. The solutions of this equation are

$$x = \frac{3 \pm \sqrt{9 - 8a^2}}{4a^2} = \frac{2}{3 \mp \sqrt{9 - 8a^2}}\,.$$

(You will actually get a cubic equation from which the trivial factor $x - 1$ can be divided out.) Show that the positive sign in the first expression for x (corresponding to the negative sign in the second one) is consistent with the relation $x \in [0, 1]$ only when $a = 1$, which is the case when $u = v = c$. In this case, $x = 1$, that is, this is the case $\eta = 0$, which, as we have already remarked, is trivial. Then, for the negative sign on the square root in the numerator, show that x lies in the range $[1/3, 1/2]$ for all values of $a \in [0, 1]$.

Verify the example mentioned in the text, in which $a = 3\sqrt{2}/5$ and $\cos(\eta) = 5/12$, showing that

$$\eta \approx 67.056553501352011261^\circ$$

and that the relative speed of O and O'' is $0.84c$

Solution: Clearing the denominator after taking $u = v$ leads to the equation

$$2u^2(1 - \cos\eta)\bigl(1 - u^2\cos\eta/c^2\bigr)^2 = 2u^2(1 - \cos\eta) - (u^4/c^2)(1 - \cos^2\eta)\,.$$

Dividing both sides by $1 - \cos\eta$, expanding the binomial to cancel the first term on the right, then dividing both sides by u^4/c^2 and rearranging, we get

$$\frac{2u^2}{c^2}\cos^2\eta - 3\cos\eta - 1 = 0\,,$$

as asserted. The quadratic formula then yields the stated solution. However, if this solution is to be the cosine of an angle, it must not be larger than 1. Now, when we take the positive sign on the square root in the denominator (the second expression given above for $\cos\eta$) we certainly have a number between 0 and 1. In fact, given that $0 \le a \le 1$, the largest value it can have is obtained by taking $a = 1$, and that value is $1/2$, corresponding to $\eta = 60^\circ$. The smallest value, corresponding to $a = 0$, is $1/3$, which corresponds to $\eta \approx 70.52877937^\circ$. If we take the negative sign on this square root, since the denominator must be at least 2, we need to have $\sqrt{9 - 8a^2} \le 1$, that is, $a \ge 1$. That value corresponds to $u = v = c$ and $\eta = 0$. In such a case, the value of w is actually indeterminate, since the denominator and numerator of the fraction that gives the value of w^2 are both 0.

It is a simple matter of computation to verify that $\eta \approx 67^\circ$ when $a = 3\sqrt{2}/5$, and that this value corresponds to a relative speed of $0.84c$.

Problem 1.12. It is well-known that any three lengths u, v, w with $u \le v \le w$ are the sides of a Euclidean triangle provided $u+v > w$. (The philosopher Immanuel Kant cited this fact as an example of what he called *synthetic a priori knowledge.*) Is this true for relativistic velocity triangles? If not, what additional conditions are needed?

Solution: The relation proved in Problem 1.9, combined with the fact that $-1 \leq \cos\eta \leq 1$ imposes restrictions that can be written as

$$\alpha\beta\Big(1 - \frac{uv}{c^2}\Big) \leq \gamma \leq \alpha\beta\Big(1 + \frac{uv}{c^2}\Big).$$

Since γ is an increasing function of w, it follows that the smallest allowable value of w for a given pair u and v corresponds to $\cos\eta = 1$, that is, an angle of 0 radians between the velocities that Y observes for X and Z. That means, they are moving in the same direction, and hence the minimum allowable w is the one given by the one-dimensional addition formula:

$$w = \frac{|u-v|}{1 - \frac{uv}{c^2}}.$$

Similarly, the maximum allowable w occurs when X and Z are traveling in opposite directions from Y, so that Y observes an angle of π radians between them:

$$w = \frac{u+v}{1 + \frac{uv}{c^2}}.$$

Problem 1.13. Show that the sum of the angles of a relativistic velocity triangle is smaller than two right angles.

Solution: Consider a triangle having sides u and v and angle η between them. Let the side opposite angle η be of length w if this is a relativistic triangle and w' if it is a Euclidean triangle. Similarly, let the angle opposite the side of length v be ξ if this is a relativistic triangle and ξ' if it is a Euclidean triangle. It will suffice to show that $\xi < \xi'$, since symmetry will then imply that $\zeta < \zeta'$ if ζ is the angle opposite the side of length u in the relativistic triangle and ζ' the corresponding angle in the Euclidean triangle. From our trigonometric formulas, we have

$$\begin{aligned}
w &= \frac{\sqrt{u^2 + v^2 - 2uv\cos\eta - u^2v^2\sin^2\eta/c^2}}{1 - uv\cos\eta/c^2},\\
w' &= \sqrt{u^2 + v^2 - 2uv\cos\eta},\\
\sin\xi &= \frac{v\sin\eta}{\alpha w(1 - uv\cos\eta/c^2)},\\
\sin\xi' &= \frac{v\sin\eta}{w'}.
\end{aligned}$$

We then find

$$\Big(\frac{\sin\xi}{\sin\xi'}\Big)^2 = \frac{(w')^2}{\alpha^2 w^2(1 - uv\cos\eta/c^2)^2}.$$

Thus, we would like to show that

$$\begin{aligned}
u^2 + v^2 - 2uv\cos\eta = (w')^2 < \alpha^2 w^2(1 - uv\cos\eta/c^2)^2 = \\
= \alpha^2(u^2 + v^2 - 2uv\cos\eta - u^2v^2\sin^2\eta/c^2).
\end{aligned}$$

Rewriting this inequality as

$$\begin{aligned}
0 &< (\alpha^2 - 1)(u^2 + v^2 - 2uv\cos\eta) - \alpha^2u^2v^2\sin^2\eta/c^2\\
&= \frac{u^2}{c^2 - u^2}(u^2 + v^2 - 2uv\cos\eta) - \alpha^2u^2v^2(1 - \cos^2\eta)/c^2\\
&= \frac{u^2}{c^2 - u^2}(u^2 + v^2 - 2uv\cos\eta) - \frac{c^2}{c^2 - u^2}u^2v^2(1 - \cos^2\eta)/c^2),
\end{aligned}$$

we can now divide through by $u^2/(c^2-u^2)$ to get the equivalent inequality

$$\begin{aligned} 0 &< u^2+v^2-2uv\cos\eta+v^2\cos^2\eta-v^2 \\ &= (u-v\cos\eta)^2 . \end{aligned}$$

This last expression is certainly positive unless $u = v\cos\eta$. But that can happen only if $\xi = \pi/2$, in other words one of the angles of the triangle is a right angle. We have already discussed this case in the text and shown that the sum of the angles is less than π in that case.

Assuming $\xi \neq \pi/2$, we have $\xi+\eta+\zeta < \xi'+\eta+\zeta' = \pi$, as required.

PROBLEM 1.14. This problem has four parts. We define the *angle defect* of a relativistic velocity triangle whose angles are ξ, η, and ζ to be the positive number $\pi-(\xi+\eta+\zeta)$. Consider a triangle with these angles and divide it into two smaller triangles by drawing a line from the vertex at angle η to a point on the opposite side, thereby dividing the triangle into two smaller triangles, one having angles η_1 (part of angle η), ξ, and φ_1 (at the vertex on the side opposite the angle η), and the other having angles η_2, ζ, and $\varphi_2 = \pi-\varphi_1$.

Part 1: Show that the defect of the original triangle is the sum of the defects of the two triangles into which it is divided. (It is not difficult to prove—although you are not being asked to do so—that when a triangle is partitioned into any number of other triangles, its defect is the sum of the defects of the triangles that partition it.) Thus the defect of a triangle is proportional to what we think of as the area of a triangle, and so we shall define the area of a triangle to be c^2 times its defect. We then define the area of a polygon to be the sum of the areas of any set of triangles into which it can be partitioned. It is not difficult to show that this definition is independent of the way in which the polygon is triangulated.

Part 2: Consider an isosceles relativistic velocity triangle having two equal sides of length u with angle η between them, and let the other two angles both be equal to ξ and the third side equal to w. Show that

$$\begin{aligned} \cos\xi &= \frac{\sin\frac{\eta}{2}}{\sqrt{1-\frac{u^2}{c^2}\cos^2\frac{\eta}{2}}}, \\ \sin\xi &= \frac{\cos\frac{\eta}{2}}{\alpha\sqrt{1-\frac{u^2}{c^2}\cos^2\frac{\eta}{2}}}, \\ w &= \frac{2u\sqrt{1-\frac{u^2}{c^2}\cos\frac{\eta}{2}}}{1-\frac{u^2}{c^2}\cos\eta}\cdot\sin\frac{\eta}{2} . \end{aligned}$$

Part 3: Consider a regular polygon P_n consisting of n isosceles triangles having vertex angle $2\pi/n$ glued together along their equal sides, which all have length u. Show that its perimeter $\pi(P_n)$ and its area $A(p_n)$, which is c^2 times its angle defect, are given by

$$\begin{aligned} \pi(P_n) &= \frac{2u\sqrt{1-\frac{u^2}{c^2}\cos^2\frac{\pi}{n}}}{1-\frac{u^2}{c^2}\cos\frac{2\pi}{n}}\cdot\Bigl(n\sin\frac{\pi}{n}\Bigr), \\ A(P_n) &= \Biggl((n-2)\pi-2n\arccos\Bigl(\frac{\sin\frac{\pi}{n}}{\sqrt{1-\frac{u^2}{c^2}\cos^2\frac{\pi}{n}}}\Bigr)\Biggr)c^2 . \end{aligned}$$

Part 4: Using the relationship

$$\lim_{\eta\to 0}\frac{\sin\eta}{\eta} = 1\,,$$

show that

$$\begin{aligned}\lim_{n\to\infty}\pi(P_n) &= 2\pi\alpha u = 2\pi u + \frac{\pi u^3}{c^2} + \frac{3\pi u^5}{8c^4} + \cdots\,,\\ \lim_{n\to\infty}A(P_n) &= 2\pi c^2(\alpha - 1) = \pi\Big(u^2 + \frac{3u^4}{8c^2} + \cdots\Big)\,.\end{aligned}$$

Thus, for small velocities u the circumference of a circle of radius u is asymptotic to the value it would have it u were a length in Euclidean space, and the same is true of the area. Equality holds if $c = \infty$.

Solution: Part 1: This is quite trivial, since of the six angles in the two smaller triangles, two (η_1 and η_2) add up to the angle η that was subdivided by the line to the opposite side, two others (φ_1 and φ_2) total exactly π, since that line creates two mutually supplementary angles where it meets the opposite side, and the remaining two (ξ and ζ) are the other two angles of the original triangle. Hence the sum of the the defects of the two smaller triangles is

$$\big(\pi - \eta_1 - \xi - \varphi_1)\big) + \big(\pi - \eta_2 - \zeta - \varphi_2\big) = 2\pi - \big(\xi + \eta + \zeta + \pi\big) = \pi - \xi - \eta - \zeta\,.$$

Part 2: The substitution $v = u$ in the trigonometric formulas yields immediately

$$\begin{aligned}\cos\xi &= \frac{1-\cos\eta}{\sqrt{2 - 2\cos\eta - \frac{u^2}{c^2}\big(1-\cos^2\eta\big)}}\\ \sin\xi &= \frac{\sin\eta}{\alpha\sqrt{2 - 2\cos\eta - \frac{u^2}{c^2}\sin^2\eta}}\,,\\ w &= \frac{u}{1-\frac{u^2}{c^2}\cos\eta}\cdot\sqrt{2(1-\cos\eta) - \frac{u^2}{c^2}(1-\cos^2\eta)}\,,\end{aligned}$$

where we have written $1 - \cos^2\eta$ in the first and last equations and the equivalent $\sin^2\eta$ in the middle one in order to simplify the computations that we are about to do.

In the first of these equations, we factor $\sqrt{2(1-\cos\eta)}$ out of the numerator and denominator, reducing it to

$$\cos\xi = \frac{\sqrt{\frac{1-\cos\eta}{2}}}{\sqrt{1 - \frac{u^2}{c^2}\frac{1+\cos\eta}{2}}}\,.$$

It is then only necessary to invoke the half-angle formulas to conclude that

$$\cos\xi = \frac{\sin\frac{\eta}{2}}{\sqrt{1-\frac{u^2}{c^2}\cos^2\frac{\eta}{2}}}\,,$$

as asserted.

In the second equation we replace the numerator by $2\sin\frac{\eta}{2}\cos\frac{\eta}{2}$, while in the denominator we replace $2(1-\cos\eta)$ by $4\sin^2\frac{\eta}{2}$ and $\sin^2\eta$ by $4\sin^2\frac{\eta}{2}\cos^2\frac{\eta}{2}$. Then,

canceling $2\sin\frac{\eta}{2}$ from numerator and denominator, we get

$$\sin\xi = \frac{\cos\frac{\eta}{2}}{\alpha\sqrt{1-\frac{u^2}{c^2}\cos^2\frac{\eta}{2}}}\,,$$

as required.

Finally, factoring out $2(1-\cos\eta) = 4\sin^2\frac{\eta}{2}$ from under the radical in the third equation and using the half-angle formulas, we get

$$w = \frac{2u\sqrt{1-\frac{u^2}{c^2}\cos^2\frac{\eta}{2}}}{1-\frac{u^2}{c^2}\cos\eta}\cdot\sin\frac{\eta}{2}\,.$$

Part 3: If you let w_n be the value of w when $\eta = 2\pi/n$, then the length of a side of P_n is in fact w_n. The formulas for $\pi(P_n)$ and $A(P_n)$ are now immediate consequences of what was just shown in Part 2.

Part 4: The relation

$$C = \lim_{n\to\infty} nw_n = 2\pi\alpha u$$

is a direct consequence of the formula for w_n proved in Part 3.

As for the area, if we let ξ_n denote the angles at the base of the isosceles triangles that constitute P_n, we observe that the formula for $\cos\xi_n$ shows that $\cos\xi_n$ tends to zero, so that ξ_n tends to $\pi/2$ as $n\to\infty$. We can therefore write

$$\lim_{n\to\infty} A(P_n) = \lim_{n\to\infty}\Big(n\big(\pi-2\xi_n\big)-2\pi\Big)c^2\,.$$

Since $\pi - 2\xi_n$ tends to zero, we have

$$\lim_{n\to\infty}\frac{\pi-2\xi_n}{\sin\big(\pi-2\xi_n\big)} \to 1\,.$$

It now follows that

$$\lim_{n\to\infty} A(P_n) = \lim_{n\to\infty}\Big(n\sin\big(\pi-2\xi_n\big)-2\pi\Big)c^2 = \lim_{n\to\infty}\big(n\sin(2\xi_n)-2\pi\big)c^2\,.$$

But this last limit is just

$$\lim_{n\to\infty}\bigg(\frac{2n\sin\frac{\pi}{n}\cos\frac{\pi}{n}}{\alpha\big(1-\frac{u^2}{c^2}\cos^2\frac{\pi}{n}\big)}-2\pi\bigg)c^2 = 2\pi(\alpha-1)c^2\,,$$

as claimed.

To get the asymptotic values for small u, all we have to do is note that

$$\alpha = \Big(1-\frac{u^2}{c^2}\Big)^{-\frac{1}{2}}$$

and invoke the binomial theorem.

Problem 1.15. Show that the sides u, v, and w of a relativistic velocity triangle given in terms of its angles are all less than c.

Solution: We recall that the side u given as

$$u = \frac{c\sqrt{\cos^2\xi+\cos^2\eta+\cos^2\zeta+2\cos\xi\cos\eta\cos\zeta-1}}{\cos\zeta+\cos\xi\cos\eta}\,.$$

To show that $u^2 \le c^2$ is simply a matter of squaring and canceling. Starting from the inequality we want, we get the following sequence of equivalent inequalities:

$$\begin{aligned} c^2(\cos^2\xi + \cos^2\eta + \cos^2\zeta + 2\cos\xi\cos\eta\cos\zeta - 1) &\le c^2(\cos\zeta + \cos\xi\cos\eta)^2\,, \\ \cos^2\xi + \cos^2\eta &\le \cos^2\xi\cos^2\eta\,, \\ \cos^2\xi(1-\cos^2\eta) &\le 1-\cos^2\eta\,. \end{aligned}$$

The last of these inequalities is obvious, and then one can reverse the steps and get back to the first one. (It is established in Problem 1.17 below that the quantity under the radical sign is positive.)

Problem 1.16. Compute the relative speed w of O and O'', given that O' has speeds $u = 4c/5$ and $v = 3c/5$ relative to them and measures the angle between their trajectories as $3\pi/4$.

Solution: The easiest thing is to use *Mathematica* Notebook 4, which reveals that

$$w = \frac{1}{553}\sqrt{205441 + 43200\sqrt{2}}\,c \approx 0.933581\,c\,.$$

Problem 1.17. Let ξ, η, and ζ be three positive angles in radian measure whose sum is less than π. Show that the expressions

$$\cos^2\xi + \cos^2\eta + \cos^2\zeta + 2\cos\xi\cos\eta\cos\zeta - 1$$

and

$$\cos\xi + \cos\eta\cos\zeta$$

are both positive, and hence that the formulas given in the text for the sides of a relativistic velocity triangle having these angles are valid.

Solution: Let φ be any angle such that $0 \le \varphi < \pi - \xi$, that is, $\xi + \varphi < \pi$. We have

$$0 = \cos\xi + \cos(\pi - \xi) < \cos\xi + \cos\varphi\,.$$

We now apply this result, taking $\varphi = \eta + \zeta$:

$$\begin{aligned} 0 &< \cos\xi + \cos(\eta+\zeta) \\ &= \cos\xi + \cos\eta\cos\zeta - \sin\eta\sin\zeta < \cos\xi + \cos\eta\cos\zeta\,. \end{aligned}$$

Next,

$$\begin{aligned} &\cos^2\xi + \cos^2\zeta + \cos^2\eta + 2\cos\xi\cos\eta\cos\zeta - 1 = \\ &\quad = (\cos\xi + \cos\eta\cos\zeta)^2 - 1 + \cos^2\eta + \cos^2\zeta - \cos^2\eta\cos^2\zeta \\ &\quad = (\cos\xi + \cos\eta\cos\zeta)^2 - (1-\cos^2\eta)(1-\cos^2\zeta) \\ &\quad = (\cos\xi + \cos\eta\cos\zeta)^2 - \sin^2\eta\sin^2\zeta \\ &\quad = (\cos\xi + \cos\eta\cos\zeta - \sin\eta\sin\zeta)(\cos\xi + \cos\eta\cos\zeta + \sin\eta\sin\zeta) \\ &\quad = (\cos\xi + \cos(\eta+\zeta))(\cos\xi + \cos(\eta-\zeta)) = (\cos\xi + \cos(\eta+\zeta))(\cos\xi + \cos(\zeta-\eta))\,. \end{aligned}$$

Since either $\eta - \zeta \ge 0$ or $\zeta - \eta \ge 0$, we can use whichever of the last two expressions is more convenient. One of them, at least provides a pair of angles ξ and $\varphi = \zeta - \eta$ or $\varphi = \eta - \zeta$ such that $0 \le \varphi < \eta + \zeta < \pi - \xi$, and so both factors in the displayed equation are positive.

PROBLEM 1.18. Suppose $(\theta, u, \varphi) \ +_L \ (\chi, v, \psi) = (\mu, w, \nu)$. Prove that for any angles d, e, and f:

$$\begin{aligned} (d+\theta, u, \varphi) \ +_L \ (\chi, v, \psi) &= (d+\mu, w, \nu)\,, \\ (\theta, u, e+\varphi) \ +_L \ (e+\chi, v, \psi) &= (\mu, w, \nu)\,, \\ (\theta, u, \varphi) \ +_L \ (\chi, v, \psi+f) &= (\mu, w, \nu+f)\,. \end{aligned}$$

Different operations are being applied to the two addends in each of these cases. Show that if we take $d = e = f$ and combine these results, we obtain a mapping $T_d(\theta, u, \varphi) = (d+\theta, u, \varphi+d)$ that satisfies the equality

$$T_d(\theta, u, \varphi) \ +_L \ T_d(\chi, v, \psi) = T_d\big((\theta, u, \varphi) \ +_L \ (\chi, v, \psi)\big)\,.$$

Thus Lorentz addition is invariant under each operation T_d. (*Caution:* This result does not enable us to reduce the dimension of the three-dimensional space we have invented to describe the addition. We cannot, for example, replace (θ, u, φ) by $(0, u, \varphi-\theta)$ and (χ, v, ψ) by $(0, v, \psi-\chi)$, even though $(\theta, u, \varphi) = T_\theta(0, u, \varphi-\theta)$ and $T_\chi(0, v, \psi-\chi) = (\chi, v, \psi)$. The difficulty is that T_θ is not the same operator as T_χ.

Solution: The first part is a very trivial computation, amounting only to the fact that $\mu = \theta + \xi$, $\nu = \psi - \zeta$, where ξ and ζ are the angles in the velocity triangle corresponding to O and O''. That accounts for the first and third relations. As for the second, if φ and χ are both increased by the amount e, then the internal angle $\eta = \varphi - \chi$ remains the same, so that the composition of the velocities will remain the same.

The other assertion follows from the chain of equalities

$$\begin{aligned} T_d(\theta, u, \varphi) \ +_L \ T_d(\chi, v, \psi) &= (d+\theta, u, d+\varphi) \ +_L \ (d+\chi, v, d+\psi) \\ &= (d+\theta, u, \varphi) \ +_L \ (\chi, v, d+\psi) \\ &= (d+\theta+\xi, w, \psi+d-\zeta) \\ &= T_d(\theta+\xi, w, \psi-\zeta) \\ &= T_d\big((\theta, u, \varphi) \ +_L \ (\chi, v, \psi)\big)\,. \end{aligned}$$

PROBLEM 1.19. Suppressing the second and third spatial dimensions, we focus attention on just the time and first spatial axes. The Lorentz transformation is

$$\begin{aligned} \tau' &= \alpha\Big(\tau - \frac{u}{c}x\Big), \\ x' &= \alpha\Big(-\frac{u}{c}\tau + x\Big). \end{aligned}$$

For a reason that will become clear in a moment, let the angle θ be

$$\theta = \arccos\left(\sqrt{1-\frac{u^2}{c^2}}\right) = \arccos\left(\frac{1}{\alpha}\right).$$

Thus, $\alpha = \sec\theta$.

Solve the second equation of the Lorentz equations for x in terms of x' and τ (and θ), then substitute the result in the first equation, so that τ' and x are expressed in terms of τ and x', yielding

$$\begin{aligned} \tau &= (\cos\theta)\tau' + (\sin\theta)x, \\ x' &= -(\sin\theta)\tau' + (\cos\theta)x. \end{aligned}$$

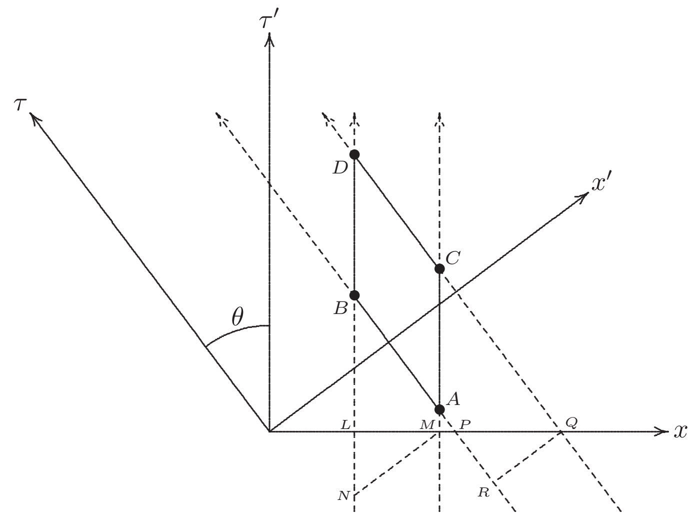

FIGURE 1. The Lorentz transformation as a tilting of time coordinates followed by a rotation, and its use in explaining the car wash paradox.

Use these equations to explain the car wash paradox.

Solution: The effect of rewriting the equations this way can be understood by imagining two new observers, one using the spatial coordinate of O and the time coordinate of O', and the other using the spatial coordinate of O' and the time coordinate of O. Between these two observers, the Lorentz transformation is a rotation through angle $\theta = \arccos(1/\alpha) = \arcsin(u/c)$.

Draw the axes with the time axis used by O (the τ-axis) perpendicular to the spatial axis used by O' (the x'-axis) and vice versa, with the two spatial axes making angle θ, where $\cos\theta = \sqrt{1 - u^2/c^2}$ is the Lorentz contraction factor, as shown in Fig. 1. The Lorentz contraction can be thought of as merely the projection of one spatial axis on the other. That interpretation makes it plain that the contraction is mutually observed: Each observer thinks the spatial axis of the other has shrunk.

Fig. 1 shows how the car wash paradox can be understood. In that figure, Observer O is the limousine driver and Observer O' is the car wash attendant. The line through AB is the world line of the front of the limousine, and the line through CD is its back. The line through AC is the front of the car wash, and the line through BD is its back. Thus the point A represents the front of the limousine entering the car wash and C represents the rear of the car entering the car wash. Simultaneous events for each observer are those that lie on a line parallel to that observer's spatial axis. Thus, for the limousine driver, the length of the car is PQ, while for the car wash attendant that length is RQ. Similarly, for the limousine driver, the length of the car wash is LM, while for the car wash attendant, it is MN. The line through B parallel to the x-axis is lower than the line through C (corresponds to a smaller value of τ, meaning that for the limousine driver event B (front of the car leaving the car wash) preceded event C (rear of the car entering the car wash). But the line through B parallel to the x'-axis corresponds to a larger

value of τ' (later time) than the parallel line through C. Hence for the car wash attendant, the rear of the car entered the car wash before the front end emerged.

PROBLEM 1.20. Here is a variation on the car wash paradox. Suppose that as the limousine moved through the car wash with speed u, two car wash attendants simultaneously, *as measured by a clock in the car wash*, put scratches in it, one in the front fender, the other in the rear fender. Suppose that they were standing 3 meters apart, *as measured by the car wash attendants themselves*, when they made the scratches. If the limousine is then stopped and measured by the car wash attendants, how far apart will the scratches be?

Solution: Without loss of generality we may choose the frames of reference so that, according to the car wash attendants, the scratch on the rear fender occurred at the origin at time 0. In other words, it was the event $(0, 0, 0, 0)$, which is also the event $(0, 0, 0, 0)$ in the frame of reference of the limousine driver, whom we assume to be moving with speed u relative to the car wash. The car wash attendants regard the scratch on the front fender as having been inflicted at time 0 at a point 3 meters ahead of the first, that is, the occurrence of this scratch is the event $(0, 3, 0, 0)$, which to the limousine driver is the event

$$\Big(-\alpha\frac{3u}{c^2}, 3\alpha, 0, 0\Big)\,.$$

Thus, the limousine driver does not think the scratches were made simultaneously. The one in front was made earlier and the distance between the two is 3α. Since the driver is at rest relative to the two scratches, this is the distance that will be measured between them when the limousine is stopped.

A shorter way of describing the situation is to say that, from the point of view of the limousine driver, the two attendants were at distance $3/\alpha$ from each other due to the relativistic contraction, and both were moving with speed $-u$. Thus, during the time between the inflicting of the two scratches the two attendants both moved a distance

$$\alpha\frac{3u^2}{c^2}$$

to the rear, so that the distance between the two scratches has to be

$$3/\alpha + \frac{3\alpha u^2}{c^2} = 3\alpha\Big(\frac{1}{\alpha^2} + \frac{u^2}{c^2}\Big) = 3\alpha\Big(\frac{1}{\alpha^2} + 1 - \frac{1}{\alpha^2}\Big) = 3\alpha\,.$$

To the car wash attendants, the distance between the scratches is 3 meters. That is the distance between the two scratches on the limousine, as observed by the car wash attendants *when the limousine is moving with speed u relative to the car wash*. But when the limousine stops moving and the two car wash attendants measure it again, the FitzGerald–Lorentz contraction of the limousine will disappear, and the distance will be 3α, exactly in agreement with what the limousine driver measured all the time. The scratches were made 3 meters apart in a contracted version of the limousine and moved apart by a factor of α when the contraction was removed.

Similarly, one could show that if the scratches were inflicted simultaneously as recorded on the dashboard clock of the limousine, then they would be $3/\alpha$ units apart when the limousine stopped, although, while it was moving, the two attendants would have judged them to be $3/\alpha^2$ units apart. (They would not have been inflicted simultaneously as measured by the clock in the car wash.)

PROBLEM 1.21. Verify that the space-time interval ds^2 between the two events—the rear of the limousine entering the car wash and the front of it leaving the car wash—is negative (spacelike).

Solution: Let the length of the limousine (at rest, or in the frame of the driver) be L. This is also the length of the car wash in the frame used by the car wash attendants. Since the space-time interval is the same for both the limousine driver and the car wash attendants, we use the car wash frame of reference to compute it. In that frame, the first event was the rear end of the limousine entering the car wash. At that instant, the front of the limousine had gone into the car wash a distance $L\sqrt{1-u^2/c^2}$. The distance that remained before the front end emerged was $L(1-\sqrt{1-u^2/c^2})$. At the speed u, then, the elapsed time between the two events was $L(1-\sqrt{1-u^2/c^2})/u$. The distance between the locations of the two events was L. Hence the space-time interval is

$$\Big(\frac{cL(1-\sqrt{1-u^2/c^2})}{u}\Big)^2 - L^2 = -\frac{2L^2\sqrt{c^2-u^2}}{c+\sqrt{c^2-u^2}} < 0\,.$$

PROBLEM 1.22. Consider the special case of a relativistic velocity triangle when $\boldsymbol{u}$ and $\boldsymbol{v}$ lie along perpendicular directions. For this case, we have $\gamma = \alpha\beta$. Recall, as noted above, that α corresponds to the cosine of the angle of rotation (A) that describes the Lorentz transformation between X and Y when they interchange time coordinates, and likewise β is the cosine of the corresponding angle (B) for the transformation between Y and Z and γ the angle (C) corresponding to the transformation between X and Z. Show that, when they are regarded as arcs on a sphere, the three angles that provide these geometric representations are the sides of a spherical right triangle in this case.

Solution: This is straightforward, since for a spherical triangle whose legs are α and β (measured by the angles they subtend at the center of the sphere), the hypotenuse is given by the spherical Pythagorean theorem:

$$\cos C = (\cos A)(\cos B)\,.$$

CHAPTER 2

Relativistic Mechanics

PROBLEM 2.1. Show that Eqs. (22)–(24) can be written in vector form as

$$\boldsymbol{v}_x(r) = \frac{1}{\alpha\eta}\boldsymbol{v}_y(s) + \frac{\alpha - 1}{\alpha\eta}\Big(\frac{\boldsymbol{v}_y(s)\cdot\boldsymbol{u}}{\boldsymbol{u}\cdot\boldsymbol{u}}\Big)\boldsymbol{u} + \frac{1}{\eta}\boldsymbol{u},$$

where $\boldsymbol{v}_x = (x^1)'\boldsymbol{i}+(x^2)'\boldsymbol{j}+(x^3)'\boldsymbol{k}$ and $\boldsymbol{v}_y = (y^1)'\boldsymbol{i}+(y^2)'\boldsymbol{j}+(y^3)'\boldsymbol{k}$ are the velocity vectors the two observers assign to the particle. Here the dot product is taken by Y. The vector $\boldsymbol{u}$ is common to the two, in accordance with our convention that a vector equation of this type is to be used only in the privileged coordinate systems where both observers take $\boldsymbol{i} = \boldsymbol{u}/u$.

Solution: The equations we are to prove say that

$$\boldsymbol{v}_x(r) = \big(\delta(y^1)'(s) + u, \alpha\delta(y^2)'(s), \alpha\delta(y^3)'(s)\big)\,.$$

The claim is that the right-hand side of this vector equation is equal to the right-hand side of the equation above. For the second and third components of the vector, that equality is equivalent to the statement that $\eta\delta = 1/\alpha^2$, which was proved in the text. It remains to verify that the first component is the same on both sides, which is to say that

$$\delta(y^1)'(s) + u = \frac{(y^1)'(s)}{\alpha\eta} + \frac{\alpha - 1}{\alpha\eta}(y^1)'(s) + \frac{u}{\eta} = \frac{1}{\eta}\big((y^1)'(s) + u\big)\,.$$

This equation in turn says that

$$\Big(\frac{1}{\eta} - \delta\Big)(y^1)'(s) = \Big(1 - \frac{1}{\eta}\Big)u = \frac{u^2}{c^2 + u(y^1)'(s)}\,(y^1)'(s)\,.$$

It suffices to verify that

$$\Big(\delta + \frac{u^2}{c^2 + u(y^1)'(s)}\Big)\eta = 1\,,$$

which is equivalent to the true equality

$$\delta\eta = 1 - \frac{u^2}{c^2} = \frac{1}{\alpha^2}\,.$$

PROBLEM 2.2. Show that $\boldsymbol{v}_y$ is constant (in s-time) if and only if $\boldsymbol{v}_x$ is constant (in r-time).

Solution: This result follows easily from the equations for acceleration in the two frames of reference:

$$\begin{aligned} x_1'' &= \frac{y_1''}{\alpha^2\eta^2}\,, \\ x_2'' &= \frac{y_2'' + \frac{u}{c^2}(y_1'y_2'' - y_1''y_2')}{\alpha^2\eta^3}\,, \\ x_3'' &= \frac{y_3'' + \frac{u}{c^2}(y_1'y_3'' - y_1''y_3')}{\alpha^2\eta^3}\,. \end{aligned}$$

It follows immediately from these three equations that if $y_i'' = 0$ for $i = 1, 2, 3$, then $x_i'' = 0$ for $i = 1, 2, 3$, and the converse result follows from the inverse set of equations.

PROBLEM 2.3. Verify that as c approaches infinity, all the equations of relativistic mechanics become the classical equations of Newtonian mechanics. In particular, show that $m_w \to m_0$ as $c \to \infty$.

Solution: As $c \to \infty$, we get $\alpha \to 1$, $\delta \to 1$, and $\eta \to 1$, and then $s = r$ and $y_1'(s) = x_1'(r) - u$, $y_2'(s) = x_2'(r)$, $y_3'(s) = x_3'(r)$, and $y_i''(s) = x_i''(r)$ for all s (for all r). The limit $m_w \to m_0$ follows trivially from the expression for m_w in terms of m_0.

PROBLEM 2.4. Use the binomial expansion

$$\Bigl(1 - \frac{v^2}{c^2}\Bigr)^{-\frac{1}{2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots$$

to verify that, as mentioned in connection with Eq. (33), the relativistic kinetic energy is

$$(m_v - m_0)c^2 = \frac{1}{2}m_0v^2 + \frac{3m_0v^4}{8c^2} + \frac{5v^6}{16c^4} + \cdots\,.$$

Deduce that, as $c \to \infty$, the relativistic kinetic energy approaches the Newtonian kinetic energy $\frac{1}{2}m_0v^2$.

Solution: The algebra involved here is immediate. Even though there is an infinite series of terms containing negative powers of c, we can appeal to the dominated convergence theorem for series to deduce that

$$\lim_{c\to\infty}(m_v - m_0)c^2 = \frac{1}{2}m_0v^2\,,$$

which is the classical kinetic energy.

PROBLEM 2.5. Show that the increase in rest mass observed by Y in the two-particle collision discussed in the text can be accounted for by saying that each particle converted the kinetic energy of the other into mass. Thus, both mass and energy are conserved in this case.

Solution: Since no external work was done on the system from Y's point of view, the total mass of the two particles now at rest relative to Y remains what it was when they were in motion, namely $m_0 + m_w = 2m_v$. In other words, the moving particles *do not* return to their original rest mass after the collision. Each has absorbed the kinetic energy of the other, and hence in Y's system each particle now has mass m_v, even though it is at rest relative to Y.

PROBLEM 2.6. Consider a particle moving along a straight line, so that both $\boldsymbol{v}$ and $\boldsymbol{a}$ also have the direction of this line. With that direction fixed, we can regard velocity, acceleration, and force as scalars. Show that in this case

$$F = \alpha^3 m_0 a\,,$$

where, as usual, $\alpha = c/\sqrt{c^2 - v^2}$.

Solution: This follows easily from the fact that for two parallel vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ with $\boldsymbol{v} = a\boldsymbol{u}$, a being a scalar, the dot product is given by $\boldsymbol{u} \cdot \boldsymbol{v} = auv$. In the present case, the formula in question for the force is

$$\boldsymbol{F} = m\Big(\boldsymbol{a} + \frac{\boldsymbol{v} \cdot \boldsymbol{a}}{c^2 - \boldsymbol{v} \cdot \boldsymbol{v}}\boldsymbol{v}\Big)\,.$$

Assuming that $\boldsymbol{v} = v\boldsymbol{i}$ and $\boldsymbol{a} = a\boldsymbol{i}$, we get $\boldsymbol{F} = F\,\boldsymbol{i}$, where

$$F = m\Big(a + \frac{va}{c^2 - v^2}\Big) = m_0\alpha a\Big(1 + \frac{v^2}{c^2 - v^2}\Big) = m_0\alpha^3 a\,.$$

PROBLEM 2.7. Prove "Newton's lemma" that the gravitational attraction exerted by a spherical shell of constant density (per unit *area*) is zero inside the sphere, while at points outside the sphere it is equal to that of a particle at the center of the sphere having mass equal to the total mass of the shell. Then show that the force exerted on a body of unit mass by a continuous, spherically symmetric mass density $\rho(r)$ (per unit *volume*) is equal to

$$\boldsymbol{F}(\boldsymbol{x}) = -\frac{4\pi G}{r^3}\bigg(\int_0^r \rho(s)s^2\,ds\bigg)\boldsymbol{x}\,,$$

where $r = |\boldsymbol{x}|$. Finally, show that the potential function $\varphi(\boldsymbol{x})$ for the force exerted on a body of unit mass is

$$\varphi(\boldsymbol{x}) = 4\pi G\int_0^r \rho(s)\Big(s - \frac{s^2}{r}\Big)\,ds\,,$$

and that

$$\nabla^2\varphi(\boldsymbol{x}) = 4\pi G\rho(r)\,.$$

Solution: We use spherical coordinates and assume that the point $\boldsymbol{x}$ is on the positive z-axis, so that $\boldsymbol{x} = (0, 0, z)$, where $z > 0$. The element of area is $dA = r^2\cos\varphi\,d\varphi\,d\theta$, where φ is latitude (measured from the equator) and θ is longitude. If the density per unit *area* is $\tilde{\rho}$, then the force per unit mass at the point $\boldsymbol{x}$ exerted by the portion of the sphere having area dA is

$$d\boldsymbol{F} = \tilde{\rho}\,G\,dA\frac{(r\cos\varphi\,\cos\theta\,, r\cos\varphi\,\sin\theta\,, r\sin\varphi - z)}{(r^2 - 2rz\sin\varphi + z^2)^{3/2}}\,.$$

and since the integrals with respect to θ over a period 2π in the first two components vanish, the total force is

$$\boldsymbol{F} = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\boldsymbol{F} = (0, 0, F)\,,$$

where

$$F = 2\pi\tilde{\rho}G\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^2\cos\varphi(r\sin\varphi - z)}{(r^2 - 2rz\sin\varphi + z^2)^{3/2}}\,d\varphi\,.$$

The substitution $x = \sin\varphi$ changes this expression into

$$F = 2\pi\tilde{\rho}r^2 G \int_{-1}^{1} \frac{rx - z}{(r^2 - 2rzx + z^2)^{3/2}}\, dx\,.$$

Finally, the substitution $u = r^2 - 2rzx + z^2$, that is, $x = (r^2 + z^2 - u)/(2rz)$, yields $rx - z = (r^2 - z^2 - u)/(2z)$ and $dx = -(1/2rz)\, du$, so that

$$\begin{aligned} F &= \frac{\pi r\tilde{\rho}G}{2z^2} \int_{(z-r)^2}^{(z+r)^2} \frac{r^2 - z^2}{u^{3/2}} - \frac{1}{u^{1/2}}\, du \\ &= \frac{\pi r\tilde{\rho}G(z^2 - r^2)}{z^2}\Big(\frac{1}{z+r} - \frac{1}{|z-r|}\Big) - \frac{\pi r\tilde{\rho}G}{z^2}\big((z+r) - |z-r|\big) \\ &= \begin{cases} -\dfrac{4\pi\tilde{\rho}r^2 G}{z^2}\,, & \text{if } \ z > r\,, \\ 0\,, & \text{if } \ z < r\,. \end{cases} \end{aligned}$$

This is Newton's result. The force due to the shell on a unit particle outside the shell at a point $\boldsymbol{z}$ at distance $z = |\boldsymbol{z}|$ from the origin is

$$-\frac{4\pi r^2\tilde{\rho}G}{z^3}\,\boldsymbol{z} = -\frac{MG}{z^3}\boldsymbol{z}\,,$$

where $M = 4\pi r^2\tilde{\rho}$ is the total mass of the shell.

For a three-dimensional radial density $\rho(r)$ per unit *volume*, we replace the two-dimensional density $\tilde{\rho}$ in the previous argument by $\rho(r)\,dr$ and then integrate on r from $r = 0$ to $r = z$ to get the gravitational force $\boldsymbol{F}(\boldsymbol{z})$ on a unit particle at $\boldsymbol{z}$. Since the matter outside the sphere about the origin containing the point $\boldsymbol{z}$ contributes nothing, we find that the force is

$$\boldsymbol{F}(\boldsymbol{z}) = -\left(\int_0^z \frac{4\pi\rho(s)s^2 G}{z^3}\, ds\right)\boldsymbol{z} = -\frac{M(z)G}{z^3}\boldsymbol{z}\,,$$

where $M(z) = \int_0^z 4\pi\rho(s)s^2\, ds$ is the total mass of the interior of the sphere containing $\boldsymbol{z}$.

Thus, as we might have anticipated, the effect is as if all the mass inside the sphere containing the point $\boldsymbol{z}$ were concentrated at the origin. We observe for later reference that $M'(z) = 4\pi\rho(z)z^2$.

If $\rho(s) = \rho$ is constant, this yields (replacing $\boldsymbol{z}$ by $\boldsymbol{r}$)

$$\boldsymbol{F}(\boldsymbol{r}) = -4\pi\rho G\left(\int_0^r s^2\, ds\right)\frac{\boldsymbol{r}}{r^3} = -\frac{GM}{r^3}\boldsymbol{r}\,,$$

where again $M = (4\pi r^3/3)\rho$ is the total mass of the ball.

Alternatively, in terms of the density,

$$\boldsymbol{F}(\boldsymbol{r}) = -(4\pi G/3)\rho\,\boldsymbol{r}\,.$$

We get the potential function $\varphi(\boldsymbol{r})$—note that φ is no longer the latitude angle!—as the integral of $-\boldsymbol{F}$ over a path $\boldsymbol{\gamma}(t)$, $0 \le t \le 1$ such that $\boldsymbol{\gamma}(0) = \boldsymbol{0}$ and $\boldsymbol{\gamma}(1) = \boldsymbol{r}$. (We are not assuming that this integral is independent of the path from $\boldsymbol{0}$ to $\boldsymbol{r}$, although it will turn out to be so. But *all* line integrals are independent

of the parametrization.) If $\gamma(t) = |\boldsymbol{\gamma}(t)|$, we have

$$\begin{aligned}\varphi(\boldsymbol{r}) &= -\int_0^1 \boldsymbol{F}\big(\boldsymbol{\gamma}(t)\big)\cdot\boldsymbol{\gamma}'(t)\,dt \\ &= \int_0^1 \frac{M\big(\gamma(t)\big)G}{\gamma(t)^3}\boldsymbol{\gamma}(t)\cdot\boldsymbol{\gamma}'(t)\,dt\,.\end{aligned}$$

If we take account of the relation $\boldsymbol{\gamma}(t)\cdot\boldsymbol{\gamma}'(t) = \gamma(t)\gamma'(t)$, which is valid for all vector-valued functions $\boldsymbol{\gamma}(t)$, we find that

$$\varphi(\boldsymbol{r}) = \int_0^1 \frac{M\big(\gamma(t)\big)G}{\gamma(t)^2}\gamma'(t)\,dt\,,$$

which tells us first that $\varphi(\boldsymbol{r})$ is a radial function $\varphi(r)$ (which is no surprise) and second that we can make the substitution $s = \gamma(t)$ in this integral and get

$$\varphi(r) = \int_0^r \frac{M(s)G}{s^2}\,ds = -\frac{M(r)G}{r} + \int_0^r \frac{M'(s)G}{s}\,ds\,.$$

The second equality here was obtained through integration by parts. Thus, since $M'(s) = 4\pi\rho(s)s^2$, we find

$$\varphi(r) = -\frac{M(r)G}{r} + 4\pi G\int_0^r s\rho(s)\,ds = 4\pi G\int_0^r \rho(s)\Big(s - \frac{s^2}{r}\Big)\,ds\,.$$

It is then a routine exercise in calculus to show that

$$\nabla^2\varphi(r) = 4\pi G\rho(r)\,.$$

CHAPTER 3

Electromagnetic Theory

Problem 3.1. Prove that the convection current density $\boldsymbol{J}_y = J_{y1}\boldsymbol{i} + J_{y2}\boldsymbol{j} + J_{y3}\boldsymbol{k}$ detected by Y is given by the equations

$$\begin{aligned} J_{y1} &= \alpha\bigl(J_{x1} - u\rho_x\bigr)\,, \\ J_{y2} &= J_{x2}\,, \\ J_{y3} &= J_{x3}\,. \end{aligned}$$

Solution: Since $\boldsymbol{v}_y$ is given in terms of $\boldsymbol{v}_x$ by the equations

$$\begin{aligned} v_{y1} &= \frac{1}{\delta}(-u + v_{x1}) \\ v_{y2} &= \frac{1}{\alpha\delta}v_{x2} \\ v_{y3} &= \frac{1}{\alpha\delta}v_{x3}\,, \end{aligned}$$

and $\rho_y = \alpha\delta\rho_x$, it is a simple matter to compute that $\boldsymbol{J}_y = \rho_y\boldsymbol{v}_y$ is given by

$$\begin{aligned} J_{y1} &= \alpha(-u + v_{x1})\rho_x = \alpha(-u\rho_x + J_{x1}) \\ J_{y2} &= J_{x2} \\ J_{y3} &= J_{x3}\,. \end{aligned}$$

Problem 3.2. Show that if $\nabla \times \boldsymbol{E} = -(1/c)\partial\boldsymbol{B}/\partial t$, then $\partial(\nabla\cdot\boldsymbol{B})/\partial t \equiv 0$. (This means that $\nabla\cdot\boldsymbol{B}$ is constant over time at each point, and hence identically zero at a given point if it vanishes at that point for even one value of t.)

Solution: We are assuming a single observer here for both sides of the equation. Hence there is no need for any subscripts. The variable t is of course time. All we need is the fact that the divergence operator $\nabla\cdot$ commutes with the operator $\partial/\partial t$. We then have

$$\frac{\partial(\nabla\cdot\boldsymbol{B})}{\partial t} = \nabla\cdot\frac{\partial\boldsymbol{B}}{\partial t} = -c\nabla\cdot(\nabla\times\boldsymbol{E}) = 0\,,$$

as asserted, since the divergence of the curl is zero. This problem shows that if Maxwell's equation for the curl of the electric field holds, then the divergence of the magnetic field is constant. Maxwell's equation for this divergence asserts that it is not only constant, but identically zero. Thus, if the curl equation holds and the divergence of $\boldsymbol{B}$ is identically zero at any point and any instant of time, then it is identically zero at that point at all times.

Problem 3.3. Show that if every "Observer Y" observes that $\nabla_y\cdot\boldsymbol{B}_y = 0$, then X will observe that

$$\nabla_x\times\boldsymbol{E}_x = -\frac{1}{c}\frac{\partial\boldsymbol{B}_x}{\partial t}.$$

Solution: The most tedious part of this problem was done in the text, and we can follow its outline in solving the problem.

We mention yet again that we are using the privileged coordinate system in which the two observers share a common first coordinate axis, and we recall that the partial derivative operators along the direction of motion for the two observers transform as:

$$\frac{\partial}{\partial y^1} = \frac{\partial x^1}{\partial y^1}\frac{\partial}{\partial x^1} + \frac{\partial r}{\partial y^1}\frac{\partial}{\partial r} = \alpha\Big(\frac{\partial}{\partial x^1} + \frac{u}{c^2}\frac{\partial}{\partial r}\Big)\,.$$

The other two partial derivative operators are the same for both observers. As in the text, we shall write t instead of r from now on.

We now assume that all observers agree on the equation $\nabla\cdot\boldsymbol{B} = 0$, that is,

$$\begin{aligned}\frac{\partial B_{y1}}{\partial y^1} + \frac{\partial B_{y2}}{\partial y^2} + \frac{\partial B_{y3}}{\partial y^3} &= 0\,,\\ \frac{\partial B_{x1}}{\partial x^1} + \frac{\partial B_{x2}}{\partial x^2} + \frac{\partial B_{x3}}{\partial x^3} &= 0\,.\end{aligned}$$

Writing $\boldsymbol{B}_y$ in terms of $\boldsymbol{E}_x$ and $\boldsymbol{B}_x$ and the partial derivative operators $\frac{\partial}{\partial y^1}$, $\frac{\partial}{\partial y^2}$, and $\frac{\partial}{\partial y^3}$ in terms of $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x^1}$, $\frac{\partial}{\partial x^2}$, and $\frac{\partial}{\partial x^3}$, we find

$$\begin{aligned}0 = \nabla_y\cdot\boldsymbol{B}_y &= \alpha\Big(\frac{\partial B_{x1}}{\partial x^1} + \frac{u}{c^2}\frac{\partial B_{x1}}{\partial t}\Big)\\ &\qquad +\alpha\Big(\frac{\partial B_{x2}}{\partial x^2} + \frac{u}{c}\frac{\partial E_{x3}}{\partial x^2}\Big) + \alpha\Big(\frac{\partial B_{x3}}{\partial x^3} - \frac{u}{c}\frac{\partial E_{x2}}{\partial x^3}\Big)\\ &= \alpha\nabla_x\cdot\boldsymbol{B}_x + \frac{\alpha u}{c}\left(\frac{1}{c}\frac{\partial B_{x1}}{\partial t} + \Big(\frac{\partial E_{x3}}{\partial x^2} - \frac{\partial E_{x2}}{\partial x^3}\Big)\right)\\ &= \frac{\alpha}{c}\left(\boldsymbol{u}\cdot\Big(\frac{1}{c}\frac{\partial \boldsymbol{B}_x}{\partial t} + \nabla_x\times\boldsymbol{E}_x\Big)\right).\end{aligned}$$

Hence if *every* "Observer Y" moving with constant velocity relative to X observes the equation $\nabla_y\cdot\boldsymbol{B}_y = 0$, then X will observe that

$$\nabla_x\times\boldsymbol{E}_x = -\frac{1}{c}\frac{\partial \boldsymbol{B}_x}{\partial t}\,.$$

PROBLEM 3.4. Derive the divergence equation for the electric field $\boldsymbol{E}$ from the curl equation for the magnetic field $\boldsymbol{B}$, assuming that at each point there is a time when $\nabla\cdot\boldsymbol{E} = 4\pi\rho$ at that point.

Solution: We start with the equation $\nabla\times\boldsymbol{B} = (1/c)\big(4\pi\boldsymbol{J} + \partial\boldsymbol{E}/\partial t\big)$, and use the fact that the divergence commutes with the time derivative, as was done above in Problem 3.2. We get

$$\begin{aligned}0 = \nabla\cdot(\nabla\times\boldsymbol{B}) &= \frac{1}{c}\Big(4\pi\nabla\cdot\boldsymbol{J} + \nabla\cdot\frac{\partial\boldsymbol{E}}{\partial t}\Big)\\ &= -\frac{4\pi}{c}\frac{\partial\rho}{\partial t} + \frac{1}{c}\frac{\partial(\nabla\cdot\boldsymbol{E})}{\partial t}\\ &= \frac{1}{c}\frac{\partial}{\partial t}\Big(-4\pi\rho + \nabla\cdot\boldsymbol{E}\Big).\end{aligned}$$

Canceling c, we get

$$0 = \frac{\partial}{\partial t}\big(-4\pi\rho + \nabla\cdot\boldsymbol{E}\big)\,,$$

which translates to

$$\nabla \cdot \boldsymbol{E} = C + 4\pi\rho\,,$$

for some constant C.

Thus, $\nabla \cdot \boldsymbol{E}$ is $4\pi\rho$ plus a quantity that is constant over time at each point. Hence if the equation holds at a given point of space at even one instant of time, then it holds at that point at all times.

Problem 3.5. Assume that the charge density and current density are zero. Show that in this case, all the components of $\boldsymbol{B}$ and $\boldsymbol{E}$ satisfy the *homogeneous three-dimensional wave equation*

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2\nabla\cdot\nabla u = c^2\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\Big)\\ &= c^2\big(\nabla^2 u\big)\,,\end{aligned}$$

where $\nabla^2 u$ is the *Laplacian* operator.

Solution: We rewrite the two Maxwell curl equations as

$$\begin{aligned}\frac{\partial \boldsymbol{E}}{\partial t} &= c\nabla\times\boldsymbol{B}\\ \frac{\partial \boldsymbol{B}}{\partial t} &= -c\nabla\times\boldsymbol{E}\,.\end{aligned}$$

Now the argument is the same in both cases, and is merely a matter of inserting one of these equations into the result of differentiating the other with respect to time. Thus, differentiating the second equation with respect to time yields

$$\begin{aligned}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} &= -\,c\nabla\times\frac{\partial \boldsymbol{E}}{\partial t}\\ &= -\,c^2\nabla\times\big(\nabla\times\boldsymbol{B}\big)\,.\end{aligned}$$

It is a straightforward computation to show that

$$\nabla\times\big(\nabla\times\boldsymbol{B}\big) = \nabla(\nabla\cdot\boldsymbol{B}) - \Big(\frac{\partial^2 \boldsymbol{B}}{\partial x^2}+\frac{\partial^2 \boldsymbol{B}}{\partial y^2}+\frac{\partial^2 \boldsymbol{B}}{\partial z^2}\Big)\,.$$

According to the Maxwell divergence equation, $\nabla\cdot\boldsymbol{B}=0$, and so this equation yields the desired result for all three components of $\boldsymbol{B}$.

Similarly, by differentiating the first equation with respect to time, we obtain

$$\begin{aligned}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} &= c^2\nabla\times\frac{\partial \boldsymbol{B}}{\partial t}\\ &= -\,c^2\nabla\times\big(\nabla\times\boldsymbol{E}\big)\,,\end{aligned}$$

and so the same argument applies.

CHAPTER 4

Precession and Deflection

Problem 4.1. Verify Eq. (63).

Solution: Once we know that $\lambda(\rho) = \ln\left(\frac{\rho+A}{\rho}\right)$, it follows from the equality $\nu = -\lambda$ that $\nu(\rho) = \ln\left(\frac{\rho}{\rho+A}\right)$, and then from the assumed form of the metric, that

$$\begin{aligned} c^2\,ds^2 &= c^2 e^{\lambda(\rho)}dt^2 - e^{\nu(\rho)}d\rho^2 - \rho^2\,d\varphi^2 - \rho^2\sin^2\varphi\,d\theta^2 \\ &= c^2\frac{\rho+A}{\rho}\,dt^2 - \frac{\rho}{\rho+A}\,d\rho^2 - \rho^2\,d\varphi^2 - \rho^2\sin^2\varphi\,d\theta^2\,. \end{aligned}$$

which is equivalent to Eq. (63).

Problem 4.2. Let us explore the "punctured disk" of non-zero relativistic velocities as a two-dimensional manifold (see Appendix 2), with a metric $ds^2 = g_{11}dx^2 + g_{12}dx\,dy + g_{21}dy\,dx + g_{22}dy^2$. We will find it easier to do this using polar coordinates (r,θ), where r ranges over the real numbers in the interval $(0,c)$ and θ is an angle in radian measure, two angles being identified as usual if they differ by 2π. To get the squared element of arc length, consider an infinitesimal triangle with vertex at the origin and two sides equal to r and $r+dr$ enclosing an angle $d\theta$. The third side will be ds. On the infinitesimal level, the squared element of arc length is

$$ds^2 = \frac{r^2 + (r+dr)^2 - 2r\,(r+dr)\,\cos d\theta - r^2(r+dr)^2\sin^2 d\theta/c^2}{(1 - r\,(r+dr)\cos d\theta/c^2)^2}\,.$$

Expand every term in the numerator in a Maclaurin series in dr and $d\theta$, using the well-known expansions $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots$ and $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x = x^2 - \frac{1}{3}x^4 + \cdots$, retaining only terms of degree 2 or less in these infinitesimals. Then expand the denominator as a geometric series—that is, using the expansion $(1-x)^{-2} = \frac{d}{dx}(1-x)^{-1} = \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) = 1 + 2x + 3x^2 + \cdots$, and multiply the two expansions together to show that

$$ds^2 = \left(1 - \frac{r^2}{c^2}\right)^{-2} dr^2 + r^2\left(1 - \frac{r^2}{c^2}\right)^{-1} d\theta^2\,.$$

Then compute the length of the radius from the origin to the point with polar coordinates (r,θ) and the circumference of the circle through that point with center at the origin. Here, r is the *coordinate* of the endpoint of the radius, not its *length*. Denote the length $R(r)$, and express the circumference of the circle as a function of R.

Solution: The computations are tedious, but routine. We simply use *Mathematica* in the obvious way to get the expression for ds^2. It is then routine to calculate the

radius $R(r)$ from the (missing) origin to the point with coordinates (r,θ). It is

$$R(r) = \int_0^r \left(1 - \frac{t^2}{c^2}\right)^{-1} dt = c\ln\sqrt{\frac{c+r}{c-r}} = c\,\operatorname{arctanh}\left(\frac{r}{c}\right).$$

Thus, the "length" R of the radius tends to infinity as $r \uparrow c$. We also have the inverse relation

$$r = c\tanh\left(\frac{R}{c}\right).$$

The circumference of the circle through $(r,0)$ is easily calculated as $2\pi r\alpha$, in perfect agreement with the result of Problem 1.14.

PROBLEM 4.3. Compute the eight Christoffel symbols Γ^k_{ij} and the Ricci tensor Ric_{ab} for the metric of the previous problem.

Solution: Taking the easy way out yet again, using *Mathematica*, we find the Christoffel symbols to be

$$\frac{2r}{c^2-r^2},\ 0,\ 0,\ -\frac{c^2+r^2}{2c^3},\ 0,\ \frac{c^2+r^2}{2r(c^2-r^2)},\ \frac{c^2+r^2}{2r(c^2-r^2)},\ 0\,.$$

The Ricci tensor, obtained the same way, is

$$\left\{\left\{\frac{c^4-6c^2r^2+r^4}{4r^2(c^2-r^2)^2}, 0\right\}, \left\{0, \frac{c^4-6c^2r^2+r^4}{4rc^3(c^2-r^2)}\right\}\right\}.$$

PROBLEM 4.4. The expression for ds^2 in Problem 4.3 was known classically as the *first fundamental form*. When the metric on a two-dimensional surface in $\mathbb{R}^3$ has symmetry ($g_{12} = g_{21}$), the element of arc length can be written as[1]

$$ds^2 = E\,dr^2 + 2F\,dr\,d\theta + G\,d\theta^2\,.$$

In that case, the element of area on the surface is

$$dA = \sqrt{EG - F^2}\,dr\,d\theta\,.$$

Compute this element of area. Then use the expression for dA to compute the area enclosed by the (punctured) circle centered at the origin passing through the point $(r,0)$. Finally, use the formula for dA to express the area of a triangle having sides u, v with included angle η.

Solution: In our case, $E = \alpha^4$ and $G = r^2\alpha^2$, so that

$$dA = r\alpha^3\,dr\,d\theta = r\left(1 - \frac{r^2}{c^2}\right)^{-3/2} dr\,d\theta\,.$$

It is then a routine computation to show that the area of the enclosed disk is

$$A = \int_0^{2\pi}\int_0^r t\left(1 - \frac{t^2}{c^2}\right)^{-3/2} dt\,d\theta = 2\pi(\alpha - 1)c^2\,,$$

and this result is again in perfect agreement with what was derived in Problem 1.14.

[1] We do apologize for the abuse of the letter G, which we have made strenuous efforts to avoid in our notation for the Ricci tensor. It appears here as a metric coefficient. The notation is due to Gauss, and seems too venerable to change.

PROBLEM 4.5. A finite piece of the hyperbolic plane can be represented accurately as part of a pseudo-sphere in $\mathbb{R}^3$ (see Appendix 1). The portion in question can be conveniently represented as the graph of a function in polar coordinates in an annulus, $k_0 < r < k$, where k_0 may be an arbitrarily small positive number:

$$\begin{aligned} z(r,\theta) &= k\left(\ln\left(\frac{k}{r}+\sqrt{\left(\frac{k}{r}\right)^2-1}\right)-\sqrt{1-\left(\frac{r}{k}\right)^2}\right) \\ &= k\left(\operatorname{arcsech}\left(\frac{r}{k}\right)-\sqrt{1-\left(\frac{r}{k}\right)^2}\right). \end{aligned}$$

With only a small amount of tedium one can compute that

$$\frac{dz}{dr} = -\sqrt{\left(\frac{k}{r}\right)^2-1}\,,$$

so that the element of arc length on this surface is very simple:

$$\begin{aligned} ds^2 &= dr^2 + r^2\,d\theta^2 + dz^2 \\ &= \left(1+\left(\frac{dz}{dr}\right)^2\right)dr^2 + r^2\,d\theta^2 \\ &= \frac{k^2}{r^2}\,dr^2 + r^2\,d\theta^2\,. \end{aligned}$$

Considering curves on the pseudo-sphere that are parameterized by arc length, show that a geodesic on which the point closest to the z-axis is $\big(r_0, \theta_0, z(r_0,\theta_0)\big)$ (assuming $k_0 < r_0$) must satisfy the system of Euler equations

$$\begin{aligned} \theta' &= \frac{r_0}{r^2} \\ rr'' - r'^2 &= \left(\frac{r_0}{k}\right)^2. \end{aligned}$$

Then show that the curve in the annulus whose polar equation is

$$r = \frac{kr_0}{\sqrt{k^2 - r_0^2(\theta-\theta_0)^2}}$$

maps to a geodesic on the pseudo-sphere.

Solution: The Euler equations are routine computations, and the first one says that

$$\frac{d}{ds}\big(2r^2\theta'(s)\big) = 0\,,$$

so that for some constant K having the physical dimension of length,

$$\theta' = \frac{K}{r^2}\,.$$

Since the parametrization is by arc length,

$$\frac{k^2}{r^2}\,(r')^2 + r^2\,(\theta')^2 = 1\,,$$

and so

$$k^2(r')^2 + K^2 = r^2\,.$$

Taking r_0 to be the smallest possible value of r on the geodesic, we get

$$K^2 = r_0^2\,,$$

since $r' = 0$ when $r = r_0$.

It is then very routine algebra to compute the second Euler equation.

To verify that the given polar equation satisfies both Euler equations, note that the equation of the curve implies

$$\frac{dr}{d\theta} = \frac{r^3}{k^3}(\theta - \theta_0)\,,$$

from which the first of the Euler equations is easily deduced by using the equation of the metric ds^2 and expressing the derivative dr/ds via the chain rule. At that point, the substitutions

$$\frac{dr}{ds} = \frac{d\theta}{ds}\,\frac{dr}{d\theta} = \frac{r_0}{r^2}\,\frac{dr}{d\theta}\,, \qquad \frac{d^2r}{ds^2} = \frac{r_0}{r^2}\,\frac{d}{d\theta}\Bigl(\frac{r_0}{r^2}\,\frac{dr}{d\theta}\Bigr)$$

render the other Euler equation equivalent to the easily proved relation

$$\frac{d}{d\theta}\Bigl(\frac{1}{r^3}\,\frac{dr}{d\theta}\Bigr) = \frac{1}{k^2}\,.$$

We remark that the quantity under the radical in the equation of the curve restricts the domain of the angular variable: $\theta_0 - k^2/r_0^2 < le\theta < \theta_0 + k^2/r_0^2$. The fact that the definition of the space requires $r < k$ imposes a slightly stronger restriction: $\theta_0 - \sqrt{k^2 - r_0^2}/r_0 < \theta < \theta_0 + \sqrt{k^2 - r_0^2}/r_0$.

Problem 4.6. Show that there is one other class of geodesics on the pseudosphere not included in the family of curves given in the previous problem, namely the curves whose parameterizations are $\bigl(r(s), \theta(s), z\bigl(r(s), \theta(s)\bigr)\bigr)$, where

$$r = r_0 e^{-s/k}\,, \quad \theta = \theta_0\,, \quad k\ln\Bigl(\frac{r_0}{k}\Bigr) < s < k\ln\Bigl(\frac{r_0}{k_0}\Bigr)\,, \quad k_0 < r_0 < k\,.$$

These are the hyperbolic analogs of lines of longitude on a sphere, and the parameter s is arc length.

Solution: This curve corresponds to a geodesic passing through the point at infinity on the pseudosphere. It makes $\theta' = 0$, so that $\theta = \theta_0$ is constant, and hence this geodesic is a line of longitude. Again, the computation of arc length is then a trivial matter of using either r or s as parameter and integrating the equation

$$ds = \frac{k}{r}\,dr\,.$$

Problem 4.7. Show that $\mathrm{Ric}_{ab} = \mathrm{Ric}_{ba}$ for the space-time metric of general relativity and that the equation $\mathrm{Ric}_{ab} = 0$ is an identity when $a \neq b$.

Solution: Obviously, the second assertion here implies the first. To verify it all we have to do is look at the explicit computation of the 16 components of the Ricci tensor. Inspection makes both results immediate. It is not recommended that this computation be attempted without a good computer algebra program. While it is within the capability of human diligence to carry out the computation, doing so is rather like trying to climb the Matterhorn on a bicycle.

$$\mathrm{Ric}_{ab} = \sum_{i=1}^{4}\left(\frac{\partial\Gamma^i_{ab}}{\partial x^i} - \frac{\partial\Gamma^i_{ai}}{\partial x^b} + \sum_{l=1}^{4}\Gamma^i_{ab}\Gamma^l_{il} - \Gamma^l_{ai}\Gamma^i_{bl}\right).$$

Problem 4.8. Show that the equations $\mathrm{Ric}_{33} = 0$ and $\mathrm{Ric}_{44} = 0$ are consequences of the equations $\mathrm{Ric}_{11} = 0 = \mathrm{Ric}_{22}$.

Solution: It was established in the text that the equations $\mathrm{Ric}_{11} = 0$ and $\mathrm{Ric}_{22} = 0$ imply that

$$\begin{aligned} \lambda(\rho) &= \ln\left(\frac{\rho - \rho_s}{\rho}\right), \\ \nu(\rho) &= -\lambda(\rho) = \ln\left(\frac{\rho}{\rho - \rho_s}\right), \end{aligned}$$

for some constant A.

Then for the system $\mathrm{Ric}_{33} = 0$ and $\mathrm{Ric}_{44} = 0$ we have the equations

$$\begin{aligned} 1 - \frac{1}{2}e^{-\nu(\rho)}\big(2 + \rho\lambda'(\rho) - \rho\nu'(\rho)\big) &= 0\,, \\ \frac{1}{2}\sin^2(\varphi)\big(2 + e^{-\nu(\rho)}(-2 - \rho\lambda'(\rho) + \rho\nu'(\rho))\big) &= 0\,. \end{aligned}$$

If we cancel $\sin^2(\varphi)$ from the second of these equations, multiply the factor of $1/2$ through the outside parentheses, and factor the negative sign out of the inner parentheses, it becomes identical to the first equation. Thus, we have only one equation to derive. Given the values of ν and λ, which imply that $\lambda'(\rho) = \frac{1}{\rho - \rho_s} - \frac{1}{\rho}$ and $\nu'(\rho) = \frac{1}{\rho} - \frac{1}{\rho - \rho_s}$, it follows that Ric_{33} is given by

$$1 - \frac{1}{2}e^{-\nu(\rho)}\big(2 - 2\rho\nu'(\rho)\big) = 1 - \frac{\rho - \rho_s}{\rho}\left(1 + \frac{\rho}{\rho - \rho_s} - 1\right) \equiv 0\,.$$

PROBLEM 4.9. In the Newtonian orbital computation, use the fact that $dr/dt = 0$ at both aphelion and perihelion to express l^2 in terms of the universal constants G and M and the planet-specific constants a and e. That is, show that $l^2 = GMa(1 - e^2)$.

Solution: The well-known equation of an ellipse with center at (h, k), axes of length $2a$ and $2b$ (where we shall assume $a \geq b$) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1\,.$$

Assuming $a \geq b$, the foci of this ellipse are at[2] $(h \pm c, k)$, where $b^2 + c^2 = a^2$. If, as is usual in the computation of a planetary orbit, we take the angle θ to be measured counterclockwise from the major axis and the origin at the focus $(h - c, 0)$, we have $h = c$, $k = 0$. Then when we change to polar coordinates $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$ (or, more accurately, $x = r\cos\theta$, $y = r\sin\theta$), we get

$$\frac{(r\cos\theta - c)^2}{a^2} + \frac{(r\sin\theta)^2}{b^2} = 1\,,$$

which, by introduction of the *eccentricity* $e = c/a$ (that is, $c = ae$), can be rewritten as

$$\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)r^2 - 2\frac{e\cos\theta}{a}r - (1 - e^2) = 0\,.$$

Since

$$\frac{1}{b^2} = \frac{1}{a^2(1-e^2)}\,,$$

[2]Here again with meet with a notational collision. The symbol c for the location of the focus of an ellipse is long established, but we need to remember that c is *not* the speed of light in this context. The contexts in which the two uses of this symbol occur are so different that there seems little danger of any confusion.

this quadratic equation in r can be rewritten as the equation

$$\frac{1-e^2\cos^2\theta}{a^2(1-e^2)}r^2 - 2\frac{e\cos\theta}{a}r - (1-e^2) = 0\,,$$

whose unique positive solution r satisfies

$$r(1-e\cos\theta) = a(1-e^2)\,.$$

(In particular, if $e=0$, this is simply a circle of radius a.)

Now we start from the differential equation

$$r'' - r(\theta')^2 = -\frac{GM}{r^2}\,,$$

and multiply by r^3, to make the term $r^4(\theta')^2 = l^2$ appear. We then divide the resulting equation by r^3 again and multiply it by $2r'$, getting

$$2r'r'' = \frac{2l^2r'}{r^3} - \frac{2GMr'}{r^2}\,.$$

If we now integrate both sides with respect to t from t_0 to t_1, where t_0 is a perihelion and t_1 an aphelion, the left-hand side integrates to $(r')^2$, which is zero at both endpoints. Thus we find that

$$0 = \Big(\frac{2GM}{r_{\text{aph}}} - \frac{l^2}{r_{\text{aph}}^2}\Big) - \Big(\frac{2GM}{r_{\text{peri}}} - \frac{l^2}{r_{\text{peri}}^2}\Big)\,,$$

and since $r_{\text{peri}} = a(1-e)$, while $r_{\text{aph}} = a(1+e)$, we find that

$$\frac{4GMe}{a(1-e^2)} - \frac{4l^2e}{a^2(1-e^2)^2} = 0\,,$$

That is,

$$l^2 = GMa(1-e^2)\,.$$

PROBLEM 4.10. Prove that $\rho^2\,(d\theta/ds)$ is constant for any radial space-time metric, that is, any metric of the form

$$ds = \sqrt{f(\rho)\,dt^2 - \frac{1}{c^2}\big(g(\rho)\,d\rho^2 - \rho^2\,d\varphi^2 - \rho^2\sin^2\varphi\,d\theta^2\big)}\,.$$

Here, ρ is again the radial space coordinate, not charge density. *Hint:* Show first that a suitable choice of coordinates allows us to take $\varphi\equiv\pi/2$, $d\varphi/ds = 0$.

Solution: As in the text, we can assume that the co-latitude coordinate φ varies in a quite small interval, and we assume that $\varphi(s_0)$ is a local minimum or maximum of this function and that the polar axis is chosen so that the value at this local extremum is $\varphi(s_0) = \pi/2$. The Euler equation for the variation, that is

$$\frac{d}{ds}\Big(\frac{\partial F}{\partial\varphi'}\Big) = \frac{\partial F}{\partial\varphi}\,,$$

where

$$F(\rho,\varphi,t',\rho',\varphi',\theta') = \sqrt{f(\rho)\,(t')^2 - \frac{1}{c^2}\big(g(\rho)\,(\rho')^2 - \rho^2\,(\varphi')^2 - \rho^2\sin^2\varphi\,(\theta')2\big)}\,,$$

says that

$$\frac{d}{ds}\Big(\frac{\rho^2\varphi'}{c^2F}\Big) = \frac{\rho^2\sin\varphi\cos\varphi}{c^2F}\big(\theta'(s)\big)^2\,.$$

Exactly as in the text, the initial-value problem for $\varphi(s)$ defined by this equation and the initial conditions $\varphi(s_0) = \pi/2$, $\varphi'(s_0) = 0$ has the constant solution $\varphi(s) = \pi/2$. Since the solution of the initial-value problem for this differential equation is unique, this is the only solution. Hence we can assume that φ has this constant value all along the geodesic. In other words, the geodesic is confined to the xy-plane. We now have to deal only with the integrand

$$G(r, t', r', \theta') = \sqrt{f(r)\,(t')^2 - \frac{1}{c^2}\Big(g(r)\,(r')^2 - r^2\,(\theta')2\Big)}\,,$$

where the radial coordinate ρ has been replaced by its planar equivalent r.

We now simply invoke Euler's variational equation for θ:

$$\frac{d}{ds}\Big(\frac{\partial G}{\partial \theta'}\Big) = \frac{\partial G}{\partial \theta} = 0\,.$$

It follows that

$$\frac{\partial G}{\partial \theta'} = \frac{-2r^2\theta'}{G} = -2r^2\frac{d\theta}{ds}$$

is constant, since $G \equiv 1$ along a geodesic. Thus, in proper time s, Kepler's second law holds and assures conservation of relativistic angular momentum.

Problem 4.11. Note the following series expansions

$$\begin{aligned}
\frac{2}{C} &= 2\Big(1 - \frac{r_s(3-e)}{a(1-e^2)}\Big)^{-\frac{1}{2}} = 2\Big(1 - \frac{3r_s}{a}\,\frac{1-e/3}{1-e^2}\Big)^{-\frac{1}{2}} \\
&= 2\bigg(1 - \frac{3r_s}{a}\Big(1 - \frac{e}{3} + e^2 - \frac{e^3}{3} + \cdots\Big)\bigg)^{-\frac{1}{2}} \\
&= 2\Big(1 + \frac{3r_s}{2a} - \frac{r_s e}{2a} + \cdots\Big) \\
\int_0^\pi (1 - m\sin^2 s)^{-\frac{1}{2}} &= \int_0^\pi \Big(1 - \frac{m}{2} + \frac{m}{2}\cos(2s)\Big)^{-\frac{1}{2}} \\
&= \int_0^\pi 1 + \frac{m}{4} - \frac{m}{4}\cos(2s)\cdots\,ds \\
m &= \frac{2r_s e}{a(1-e^2) - r_s(3-e)} = \frac{2r_s e}{a(1-e^2)} + \cdots\,.
\end{aligned}$$

By neglecting terms that are not larger than $r_s e/a$—since r_s/a is already very small, as is the the eccentricity $e = 0.205$, so that $e/3$ is less than 7%—show that the increment in the polar angle θ when the angle φ increases by 2π is approximately

$$\Delta\theta = 2\pi + \frac{3\pi r_s}{a(1-e^2)} = 2\pi + \frac{24\pi^3 a^2}{T^2 c^2 (1-e^2)}\,,$$

as Einstein asserted.

Solution: Since $3r_s/a$ is already extremely small, we can certainly afford to neglect 7% of it. Using the indicated series, truncated by discarding all terms that are less than 10% of the leading term $\frac{3r_s}{a}$, we find that

$$\Delta\theta = \int_0^\pi 2\Big(1 - \frac{3r_s}{2a}\Big)\,ds = 2\pi + \frac{3\pi r_s}{a}\,,$$

as asserted.

PROBLEM 4.12. For a metric that has symmetry, that is, $g_{ij} = g_{ji}$, it is trivial to show that the Christoffel symbols are also symmetric in the two subscripts. Show that in this case, the metric coefficients g_{ij} satisfy the system of first-order linear partial differential equations

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{li} \,.$$

It follows that the metric coefficients can be determined from the values they have at any one point, provided the Christoffel coefficients are given.

Solution: By inserting the values of the Christoffel coefficients on the right-hand side of this equation, we find that

$$\begin{aligned} \Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{li} &= \frac{1}{2} g_{lj} g^{lm} \Big(\frac{\partial g_{im}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^m} \Big) \\ &\quad + \frac{1}{2} g_{li} g^{lm} \Big(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \Big) \,. \end{aligned}$$

When these sums are collapsed using the relations $g_{lj} g^{lm} = \delta^m_j$ and $g_{li} g^{lm} = \delta^m_i$, the result is the relation

$$\Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{li} = \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{ij}}{\partial x^k} \,.$$

PROBLEM 4.13. Use the result of the last problem to establish the dual differential equation

$$\frac{\partial g^{ij}}{\partial x^k} = -\big(\Gamma^j_{mk} g^{im} + \Gamma^i_{mk} g^{mj} \big) \,.$$

Solution: Let g be the matrix of metric coefficients, that is, $g = (g_{ij})$. The relation $I = g^{-1} g$ implies that

$$\frac{\partial g^{-1}}{\partial x^k} = -g^{-1} \frac{\partial g}{\partial x^k} g^{-1} \,.$$

In terms of matrix entries, this equation says

$$\frac{\partial g^{ij}}{\partial x^k} = -g^{il} \frac{\partial g_{lm}}{\partial x^k} g^{mj} \,,$$

which, by the previous problem says

$$\begin{aligned} \frac{\partial g^{ij}}{\partial x^k} &= -g^{il} \big(\Gamma^p_{lk} g_{pm} + \Gamma^p_{mk} g_{pl} \big) g^{mj} \\ &= -\big(g^{il} \Gamma^j_{lk} + \Gamma^i_{mk} g^{mj} \big) \\ &= -\big(\Gamma^j_{mk} g^{im} + \Gamma^i_{mk} g^{mj} \big) \,. \end{aligned}$$

The last equation follows since the index l is a dummy index of summation and can be replaced by the letter m.

PROBLEM 4.14. Again assuming symmetry of the metric coefficients, show that if $s \mapsto \big(x^1(s), \ldots, x^n(s) \big) = \boldsymbol{\gamma}(s)$ is the parametrization of a path that minimizes arc length with the parameter s being arc length s, then $\boldsymbol{\gamma}$ satisfies the system of differential equations

$$(x^m)''(s) + \Gamma^m_{jk}\big(x^1(s), \ldots, x^n(s) \big) (x^j)'(s) (x^k)'(s) = 0 \,,$$

for $m = 1, 2, \ldots, n$.

Solution: Let

$$F(x^1,\dots,x^n,(x^1)',\dots,(x^n)') = \sqrt{g_{ij}(x^1,\dots,x^n)(x^i)'(x^j)'}\,.$$

Then, for a geodesic, we have

$$F\bigl(x^1(s),\dots,x^n(s),(x^1)'(s),\dots,(x^n)'(s)\bigr) \equiv 1\,,$$

so that

$$\frac{d}{ds}F\bigl(x^1(s),\dots,x^n(s),(x^1)'(x),\dots,(x^n)'(s)\bigr) \equiv 0\,.$$

Euler's equation for a geodesic says that

$$\frac{d}{ds}\Bigl(\frac{\partial F}{\partial (x^k)'}\Bigr) = \frac{\partial F}{\partial x^k}\,,\quad k = 1,2,\dots,n\,.$$

Now

$$\frac{\partial F}{\partial x^k} = \frac{1}{2F}\frac{\partial g_{ij}}{\partial x^k}(x^i)'\,(x^j)'\,,$$

and

$$\frac{\partial F}{\partial (x^k)'} = \frac{1}{2F}\Bigl(g_{kj}(x^j)' + g_{ik}(x^i)'\Bigr) = \frac{1}{F}g_{ik}(x^i)'\,.$$

Since $(d/ds)(1/F) \equiv 0$, it follows that

$$\frac{d}{ds}\Bigl(\frac{\partial F}{\partial (x^k)'}\Bigr) = \frac{1}{F}\Bigl(\frac{\partial g_{ik}}{\partial x^j}\,(x^i)'\,(x^j)' + g_{ik}(x^i)''\Bigr)\,.$$

Thus we have

$$\frac{\partial g_{ik}}{\partial x^j}\,(x^i)'\,(x^j)' + g_{ik}(x^i)'' = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\,(x^i)'\,(x^j)'\,.$$

But the identity

$$\frac{\partial g_{ik}}{\partial x^j}\,(x^i)'\,(x^j)' = \frac{1}{2}\Bigl(\frac{\partial g_{ik}}{\partial x^j}\,(x^i)'\,(x^j)' + \frac{\partial g_{jk}}{\partial x^i}\,(x^i)'\,(x^j)'\Bigr)$$

is a trivial matter, since the summation indices are dummies and can be interchanged with no change in the value of the formula.

Thus we find that

$$g_{ik}(x^i)'' + \frac{1}{2}\Bigl(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\Bigr)\,(x^i)'\,(x^j)' = 0\,.$$

If we now multiply these equations by g^{mk} and sum over k, we get

$$(x^m)'' + \Gamma^m_{ij}\,(x^i)'\,(x^j)' = 0\,,$$

as asserted.

Problem 4.15. Show that if the arc length s in the previous problem is replaced by an arbitrary parameter t for which $ds/dt > 0$ at all points, the equation satisfied is

$$(x^m)''(t) - \frac{d^2s}{dt^2}\,\frac{dt}{ds}\,(x^m)'(t) + \Gamma^m_{jk}\bigl(x^1(t),\dots,x^n(t)\bigr)(x^j)'(t)\,(x^k)'(t) = 0\,.$$

Notice that this equation is the same as the one derived in the previous problem if s is a *linear* function of t.

Solution: This is simply a matter of expressing the second derivative with respect to s in terms of derivatives with respect to t:

$$\begin{aligned}
\frac{d^2x^m}{dt^2} &= \frac{ds}{dt}\frac{d}{ds}\Big(\frac{dx^m}{dt}\Big) \\
&= \frac{ds}{dt}\bigg(\frac{d}{ds}\frac{ds}{dt}\Big(\frac{dx^m}{ds}\Big)\bigg) \\
&= \frac{ds}{dt}\frac{d}{ds}\Big(\frac{ds}{dt}\Big)\frac{dx^m}{ds} + \Big(\frac{ds}{dt}\Big)^2\frac{d^2x^m}{ds^2} \\
&= \frac{d}{ds}\Big(\frac{ds}{dt}\Big)\frac{dx^m}{dt} + \Big(\frac{ds}{dt}\Big)^2\frac{d^2x^m}{ds^2}\,.
\end{aligned}$$

We then have

$$\frac{d}{ds}\frac{ds}{dt} = \frac{dt}{ds}\frac{d^2s}{dt^2}\,,$$

and thus

$$(x^m)''(s) = \Big(\frac{dt}{ds}\Big)^2(x^m)''(t) - \Big(\frac{dt}{ds}\Big)^3\frac{d^2s}{dt^2}(x^m)'(t)\,.$$

By the chain rule,

$$\Gamma^m_{ij}\,(x^i)'(s)\,(x^j)'(s) = \Big(\frac{dt}{ds}\Big)^2\Gamma^m_{ij}\,(x^i)'(t)\,(x^j)'(t)\,.$$

Canceling the common factor $(dt/ds)^2$ from the whole equation now gives the desired result.

Problem 4.16. Show that the Christoffel symbols are not altered (and hence neither is the Ricci tensor) if each of the metric coefficients g_{ij} is multiplied by the same constant k. Thus, curvature is independent of the scale by which distances are measured, as we should hope if it is to be the same number for all observers.

Solution: Multiplying g_{ij} by a constant k results in each of its partial derivatives being multiplied by the same constant, while the entries g^{ij} in the inverse matrix are multiplied by $1/k$. As a result, Γ^i_{jk} remains unaltered. Since the Ricci tensor is entirely determined by the Christoffel symbols, it also remains unaltered.

Problem 4.17. Show that there is no surface $z = f(r,\theta)$ in $\mathbb{R}^3$ for which the metric coefficients induced from the metric on $\mathbb{R}^3$ are given by the diagonal matrix

$$\begin{pmatrix} \frac{r}{r+r_s} & 0 \\ 0 & r^2 \end{pmatrix}.$$

By the phrase "induced from the metric on $\mathbb{R}^3$," we mean that $ds^2 = dr^2 + r^2\,d\theta^2 + dz^2 = dr^2 + r^2\,d\theta^2 + \Big(\partial f/\partial r)\,dr + (\partial f/\partial\theta)\,d\theta\Big)^2 = \big(1 + (\partial f/\partial r)^2\big)\,dr^2 + 2(\partial f/\partial r)\,(\partial f/\partial\theta)\,dr\,d\theta + \big((r^2 + (\partial f/\partial\theta)^2\big)\,d\theta^2$.

Solution: From the coefficient of $d\theta^2$, we see that f would depend only on r. We would then have the equation

$$1 + \Big(\frac{\partial f}{\partial r}\Big)^2 = \frac{r}{r+r_s}\,,$$

so that

$$\Big(\frac{\partial f}{\partial r}\Big)^2 = \frac{r}{r+r_s} - 1 < 0\,,$$

which is impossible.

PROBLEM 4.18. Show that a non-constant uniformly almost-periodic function of time has more than one local extreme value. The definition of uniform almost-periodicity is as follows: A complex-valued function $f(t)$ of a real-variable is uniformly almost-periodic if for every $\varepsilon > 0$ there is a length L_ε such that every interval of length at least L_ε contains an *ε-translate*, which is a number T such that

$$|f(t+T) - f(t)| < \varepsilon$$

for all real numbers t. The fundamental theorem of almost-periodic functions says that if $f(t)$ is such a function, then for every $\varepsilon > 0$ there is a finite generalized trigonometric polynomial

$$p(t) = \sum_{j=1}^{n} c_{\lambda_j} e^{i\lambda_j t}$$

such that $|f(t) - p(t)| < \varepsilon$ for all t. The terms in this finite sum represent Ptolemy's epicycles. The frequencies λ_j are arbitrary, so that these polynomials are generally *not* periodic.

Solution: Let $t = a$ and $t = b > a$ be two times at which $f(b) > f(a)$. Let $\varepsilon = f(b) - f(a)$. Let $L > b - a$ be a length such that every interval of length at least L contains at least one $\varepsilon/3$-translate of the function $f(t)$. Then we can find such a translate T_0 in particular between $a + L$ and $a + 2L$. Then, in particular $a + T_0 > a + L > b$ and $f(a + T_0) \le f(a) + |f(a + T_0) - f(a)| < f(a) + \varepsilon/3$. And similarly $b+T_0 > a+T_0$, so that $f(b+T_0) \ge f(b) - |f(b+T_0) - f(b)| > f(b) - \varepsilon/3 > f(a) + \varepsilon/3 > f(a + T_0)$. Then the maximum value of the function on the interval $[a, a+T_0]$ is at least $f(b)$, and is not attained at either endpoint, since the endpoint values are both less than $f(b)$. Similarly, the minimum value of $f(t)$ on the interval $[b, b + T_0]$ is at most $f(a + T_0)$. Again, it is not attained at either endpoint. Thus there is a local maximum $t_1 \in [a, a+T_0]$ and a local minimum $t_2 \in [b, b+T_0]$. These points are not the same, since $f(t_1) \ge f(b) > f(a + T_0) > f(t_2)$.

PROBLEM 4.19. Finish the proof of Theorem 4.3.

Solution: If u_1 is a local minimum, then $D > 0$. Then, if $\beta > 0$, the quadratic in w has either two positive roots or no positive roots. In the latter case, since the cubic polynomial has a positive leading coefficient, it is negative for all $u_2 < u < u_1$, where u_2 is the next smaller root (if any) and $u_2 = -\infty$ if u_1 is the only root. In any case, $u(\theta)$ cannot assume values in the range $u_2 < u < u_1$, since they would make p^2 negative. Thus $u \ge u_1$ for all θ and the function u, if not constant, must increase continually. without bound, since it cannot reach any maximum value, where p would vanish. In fact, it cannot even remain bounded in this case, since its derivative is also ultimately an increasing function.

Finally, if $\beta < 0$, then the quadratic in w has one positive root u_2, and the cubic polynomial in u, whose leading coefficient is now negative, has positive values for $u_1 < u < u_2$. The function $u(\theta)$ is therefore increasing at u_1. If it reaches the maximum u_2 at a finite value of θ, it will then begin to decrease again, and thereafter oscillate between u_1 and u_2.

PROBLEM 4.20. Is it possible to endow space-time with a metric of the form

$$ds^2 = f(\rho)\, dt^2 - \frac{1}{c^2}\big(g(\rho)\, d\rho^2 + \rho^2\, d\varphi^2 + \rho^2 \sin^2\varphi\, d\theta^2\big)$$

such that the following conditions are met?

(1) $f(\rho) \to 1$ and $g(\rho) \to 1$ as $\rho \to \infty$. (That is, the metric approaches the flat-space metric at infinity.)
(2) The equations of a geodesic given by the Euler equations are Newton's equations of motion (52) and (53).

Hint: Notice the quotation from Eddington at the beginning of Section 4.4.

Solution: The problem requires some clarification, since the independent variable in Newton's equations is time, but we have two sorts of time in this metric: the observed laboratory time and the proper time on the geodesic. We know that Eq. (53) does hold with proper time s replacing laboratory time t. Consequently, it cannot hold with t as the independent variable. The remaining question is whether we can get Eq. (52) for the geodesic with s replacing t under the given requirements. In fact, we cannot.

The Euler equation on φ, as before, reduces the problem via a rotation of axes to a planar problem; and then the Euler equations on t and θ once again yield $dt/ds = k/(2f(r))$ and $d\theta/ds = l/r^2$, where k is a dimensionless constant and l is angular momentum. The equation on r coupled with the metric equation then implies

$$\frac{d^2r}{ds^2} = -\frac{c^2k^2}{8f(r)g(r)}\Big(\frac{f'(r)}{f(r)} + \frac{g'(r)}{g(r)}\Big) + \frac{c^2g'(r)}{2(g(r))^2} + \frac{l^2}{2r^2g(r)}\Big(\frac{g'(r)}{g(r)} + \frac{2}{r}\Big).$$

The right-hand side of this equation needs to be

$$-\frac{GM}{r^2} + r\Big(\frac{d\theta}{ds}\Big)^2 = -\frac{GM}{r^2} + \frac{l^2}{r^3}.$$

Thus, comparing the two terms that contain the factor l^2, we have the equation

$$\frac{1}{r} = \frac{1}{2g(r)}\Big(\frac{g'(r)}{g(r)} + \frac{2}{r}\Big).$$

The general solution of this equation is

$$g(r) = \frac{1}{1 - Lr^2},$$

and then the requirement that $g(r) \to 1$ as $r \to \infty$ means that $L = 0$. That is, $g(r) \equiv 1$. We are then left with the equation

$$\frac{GM}{r^2} = \frac{1}{8}c^2k^2\frac{f'(r)}{(f(r))^2}.$$

whose general solution is

$$\frac{c^2k^2}{8f(r)} = K + \frac{GM}{r}.$$

The condition $f(r) \to 1$ as $r \to \infty$ implies that

$$K = \frac{c^2k^2}{8},$$

so that

$$f(r) = \frac{r}{r + r_1},$$

where $r_1 = 8GM/c^2k^2$, which is $4/k^2$ times the Schwarzschild radius r_s.

We now have Newton's equations of motion with t replaced by s. But there is an apparent flaw in the derivation, since r_1 depends on the constant k, which

is—at least *a priori*—specific to each particular orbit, not a constant that can be determined knowing only the physical parameters G, M, and c. That is, this constant cannot be incorporated into the general metric. This flaw is ineradicable, as we now show by consideration of circular orbits.

Fix a circular orbit of radius r, so that $0 = dr/ds = d^2r/ds^2$, and by Newton's equation

$$\Bigl(\frac{d\theta}{ds}\Bigr)^2 = \frac{GM}{r^3}\,.$$

On the other hand, for circular orbits we have

$$1 = \frac{r}{r+r_1}\Bigl(\frac{dt}{ds}\Bigr)^2 - \frac{r^2}{c^2}\Bigl(\frac{d\theta}{ds}\Bigr)^2\,,$$

so that

$$\begin{aligned} 1 &= \frac{k^2(r+r_1)}{4r} - \frac{GM}{c^2 r} \\ &= \frac{k^2}{4} + \frac{r_s}{2r}\,, \end{aligned}$$

and

$$k^2 = 4 - \frac{2r_s}{r}\,,$$

and thus k is *not* independent of the particular orbit. We do not have a single parameter r_1 determined by the mass of the particle at the origin, but a different parameter for each orbit. In contrast, the Schwarzschild radius $r_s = 2GM/c^2$ that appears in the relativistic solution is the same for all orbits, being determined entirely by the mass of the particle at the origin.

PROBLEM 4.21. Derive Eqs. (49) and (50) by using a moving frame of reference $\boldsymbol{\omega}_1(\theta) = \cos\theta\,\boldsymbol{i} + \sin\theta, \boldsymbol{j}$, $\boldsymbol{\omega}_2 = -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}$ and setting the dot products of $(\alpha\boldsymbol{r}')' + (GM/r^3)\boldsymbol{r}$ with $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ equal to zero. Solve the second of these equations for $\theta''(t)$ and substitute that value in the first of them. If you try to do this without the assistance of *Mathematica*, keep in mind that a fraction vanishes if and only if its numerator vanishes.

Solution: It really is pointless not to make use of *Mathematica* for this work. Here is a quick-and-dirty sequence of commands that yields the expression that needs to be equated to zero. No attempt has been made to achieve any elegance at all. We begin by defining $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, as suggested. We then follow the suggested program. But we start with a function $\sigma(t) = s(t)\bigl(\cos(\theta(t)), \sin(\theta(t))\bigr)$, using the letter s instead of the usual r. That is because we need to replace $r'(t)$ by $r'(t)\theta'(t)$, and that is easier to do, if $r'(t)$ is called $s'(t)$.

```
ω1[t_] := {Cos[θ[t]], Sin[θ[t]]};
ω2[t_] := {-Sin[θ[t]], Cos[θ[t]]};
σ[t_] := s[t] {Cos[θ[t]], Sin[θ[t]]};
α[t_] := (1 - Dot[σ'[t], σ'[t]]/c^2)^(-1/2);
FullSimplify[Dot[D[α[t] σ'[t],t] + (G M α[t]/Dot[σ[t],
    σ[t]]^(3/2)) σ[t], ω2[t]]]
a = % /.  {s[t] -> r[θ], s'[t] -> r'[θ] θ'[t],
    s''[t] -> r''[θ] θ'[t]^2 + r'[θ] θ''[t]}
```

```
Solve[a == 0, θ''[t]]
b = θ''[t] /.  %[[1]]
FullSimplify[b]
d = FullSimplify[Dot[D[α[t] σ'[t],t] + (G M α[t]/Dot[σ[t],
    σ[t]]^(3/2))σ[t], ω1[t]]]
f = FullSimplify[d] /.  {s[t] -> r[θ], s'[t] -> r'[θ] θ'[t],
    s''[t] -> r''[θ] θ'[t]^2 + r'[θ] θ''[t]}
Simplify[f, r[θ] > 0]
g = % /.  θ''[t] -> b
h = FullSimplify[%]
Numerator[h]
```

The output from this last command is

```
G M - r[θ] θ'[t]^2 (r[θ]^2 + 2 r'[θ]^2 - r[θ] r''[θ])
```

Setting this expression equal to zero is tantamount to Eq. (49), as required.

The factor $\theta'(t)$ that occurs here might appear to be troublesome. However, we recall that $\theta' = l/(\alpha r^2)$, where l is the relativistic angular momentum, which is constant. Since θ' also occurs in the expression for α, we get the useful equation

$$\alpha^2 = 1 + \frac{l^2}{c^2}\Big(\frac{1}{r^2} + \frac{\left(\frac{dr}{d\theta}\right)^2}{r^4}\Big)\,,$$

and then when we pass to the variable $u = 1/r$, this simply becomes $\alpha^2 = 1 + (l^2/c^2)\big(u^2 + (u')^2\big)$. That, at last, yields Eq. (50).

Problem 4.22. Show that in Newtonian mechanics, the constant angular momentum per unit mass of a planet is $l = 2\sqrt{GMa(1-e^2)}$, where a is the average distance of the planet from the sun and e is the eccentricity of the elliptic orbit.

Solution: The angular momentum per unit mass is twice the rate at which area is swept out. That rate is the area of the ellipse divided by the period, which is to say

$$\frac{2\pi ab}{\frac{2\pi a^{3/2}}{\sqrt{GM}}} = \frac{a\sqrt{GM(1-e^2)}}{\sqrt{a}} = \sqrt{GMa(1-e^2)}\,.$$

Here we used the fact that the semi-minor axis b is $a\sqrt{1-e^2}$.

Problem 4.23. Show that the solution of Eq. (50) with $u(0) = u_0$, $u'(0) = 0$ is the inverse of the function $\theta = \theta(u)$ given by

$$\theta = \int_{u_0}^{u} \frac{dx}{\sqrt{q(u_0)e^{2s(x-u_0)} - q(x)}}\,,$$

where $q(x) = \big(x^2 + 2p/r_s)\big)$, $p = GM/l^2$, and $r_s = 2GM/c^2$.

Show also that in this case we have

$$u'' + u = \frac{1}{2}q(u_0)e^{2r_s(u-u_0)}\,.$$

Thus, the study of Eq. (50) is subsumed in the general study of equations of the form

$$u'' + u = ae^{bu}\,,$$

with constants a and b.

Solution: Let $v(u)$ be such that $u'(\theta) = \sqrt{v\big(u(\theta)\big)}$. We then calculate easily that $u''(\theta) = v'\big(u(\theta)\big)/2$, and our equation now appears to be (suppressing the argument θ)

$$v'(u) + 2u = 2p + r_s\big(u^2 + v\big)\,,$$

that is,

$$v' - r_s v = 2p + r_s u^2 - 2u = r_s q(u) - q'(u)\,.$$

This is a linear equation for v, and $e^{-r_s u}$ is an integrating factor for it. The standard techniques of elementary differential equations then imply that for some constant K,

$$v(u) = Ke^{r_s u} - q(u)\,,$$

where q is the quadratic polynomial given above. The fact that $v(u_0) = 0$ then gives the value of K as $q(u_0)e^{-r_s u_0}$, and the expression for $u'' + u$ is then a routine computation.

Remark 4.1. Although it is great fun to play with differential equations like this one, we are compelled to admit that this one doesn't correspond to reality very well. It implies that the right-ascension angle measured from u_0 (the aphelion value) to any value of u is

$$\theta = \int_{u_0}^{u} \frac{dx}{\sqrt{q(u_0)e^{r_s(x-u_0)} - q(u)}}\,,$$

and when it is adjusted for Mercury, with $p = GM/l^2 = 1/\big(a(1-e^2)\big)$ and $r_s = 2953.8$ m, the orbital data $a = 5.7909 \times 10^{10}$ m, $u_0 = 1/(6.98169 \times 10^{10}$ m), $e = 0.20563$, and perihelion value $u_1 = 1/(4.60012 \times 10^{10}$ m), numerical integration implies that the increase in the right-ascension angle between aphelion and perihelion is 3.137 radians, about one 250^{th} of a radian (one-fifth of a degree) short of what it should be, namely π. If it were this amount in excess, we might take some comfort from the fact that the equation predicts a precession of the perihelion. Unfortunately, however, this precession is in the direction opposite to the orbital motion, and hence is at variance with reality.

Problem 4.24. Assuming Newtonian mechanics, imagine that a photon "falls" from infinity with initial speed c, use conservation of energy to show that its speed v at distance r from the sun satisfies

$$v^2 = c^2 + 2GM/r\,,$$

that is, $v = c\sqrt{1 + r_s/r}$. (In contrast to what we found about the speed of light in a relativistic gravitational field, Newtonian mechanics predicts that light slows down as r increases. At the Schwarzchild radius, the speed of light would be $\sqrt{2}c$. In the case of the sun, however, where $r_s/r < 10^{-5}$ for any photon that does not actually fall into the sun, the speed would be nearly constant, and the resulting path nearly straight.)

Solution: This being Newtonian mechanics, we can assume that light has mass and is subject to gravity, or at least, we can ascribe to it a fictitious kinetic energy "per unit mass" equal to $c^2/2$ and potential energy per unit mass equal to GM/r, as for

any other body in a gravitational field. Conservation of energy then says that if v is the speed of light at distance r, we must have

$$\frac{1}{2}c^2 = \frac{1}{2}v^2 - \frac{GM}{r} = \frac{1}{2}v^2 - \frac{r_s}{r}c^2\,.$$

CHAPTER 5

Concepts of Curvature, 1700–1850

PROBLEM 5.1. Verify that the curvature of a circle of radius r_0 is $1/r_0$.

Solution: Parameterize the circle as $\boldsymbol{r}(t) = r_0 \cos(t)\boldsymbol{i} + r_0 \sin(t)\boldsymbol{j}$. Then the element of arc length is

$$ds = |\boldsymbol{r}'(t)|\,dt = r_0\,dt\,.$$

It follows that if we simply take $t = s/r_0$, so that $\boldsymbol{r}(s) = r_0 \cos(s/r_0)\boldsymbol{i} + r_0 \sin(s/r_0)\boldsymbol{j}$, we get a unit tangent vector

$$\boldsymbol{r}'(s) = -\sin(s/r_0)\boldsymbol{i} + \cos(s/r_0)\boldsymbol{j}\,,$$

whose derivative with respect to arc length is

$$\boldsymbol{r}''(s) = -\frac{1}{r_0}(\cos(s/r_0)\boldsymbol{i} + \sin(s/r_0)\boldsymbol{j}\,.$$

and that $|\kappa(s)|$ has the constant value $1/r_0$. Since the ordered pair of unit vectors $\bigl(\boldsymbol{v}(s), \boldsymbol{a}(s)\bigr)$, where $\boldsymbol{v}(s) = \boldsymbol{r}'(s) = -\sin(s/r_0)\boldsymbol{i} + \cos(s/r_0)\boldsymbol{j}$ and $\boldsymbol{a}(s) = -\cos(s/r_0)\boldsymbol{i} - \sin(s/r_0)\boldsymbol{j}$, is a right-handed system, the curvature in this case is positive.

PROBLEM 5.2. Assume that the tangent plane to the surface $z = f(x, y)$ (where $f(0, 0) = 0$) is horizontal at the point $(0, 0, 0)$, so that $\partial f/\partial x = 0 = \partial f/\partial y$ at this point. It was shown in the text that the curvature at the point $(0, 0, 0)$ is the determinant of the Hessian, that is,

$$\kappa(0,0) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \Bigl(\frac{\partial^2 f}{\partial x\,\partial y}\Bigr)^2.$$

Assume now that the coordinates (x, y, z) are changed by an orthogonal matrix

$$O = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

into the coordinates (u, v, w), and that in the new coordinates the equation of the surface is $w = h(u, v)$. That is,

$$\begin{aligned} u &= a_{11}x + a_{12}y + a_{13}f(x,y)\,, \\ v &= a_{21}x + a_{22}y + a_{23}f(x,y)\,, \\ w = h(u,v) &= a_{31}x + a_{32}y + a_{33}f(x,y)\,. \end{aligned}$$

Show that in the new coordinates the curvature is given as

$$\kappa(0,0) = \frac{\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x\,\partial y})^2}{\bigl(1 + (\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2\bigr)^2}\,.$$

Solution: We note first that the expression for the curvature is unaffected by a rotation about the z-axis, that is, if (x, y) is replaced by $\big((\cos\theta)s-(\sin\theta)t, (\sin\theta)s+(\cos\theta)t\big)$, which is equivalent to $(s,t) = \big((\cos\theta)x + (\sin\theta)y, -(\sin\theta)x + (\cos\theta)y\big)$. For the new expression $f(x,y) = f\big((\cos\theta)s - (\sin\theta)t, (\sin\theta)s + (\cos\theta)t\big) = g(s,t) = g\big((\cos\theta)x + (\sin\theta)y, -(\sin\theta)x + (\cos\theta)y\big)$ satisfies

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial g}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial g}{\partial t}\frac{\partial t}{\partial x} = \cos\theta\frac{\partial g}{\partial s} - \sin\theta\frac{\partial g}{\partial t}\\
\frac{\partial f}{\partial y} &= \frac{\partial g}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial g}{\partial t}\frac{\partial t}{\partial y} = \sin\theta\frac{\partial g}{\partial s} + \cos\theta\frac{\partial g}{\partial t}\,.
\end{aligned}$$

These equations imply that the partial derivatives of g vanish at the origin, since those of f vanish there. We also have

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \cos^2\theta\frac{\partial^2 g}{\partial s^2} - 2\cos\theta\sin\theta\frac{\partial^2 g}{\partial s\,\partial t} + \sin^2\theta\frac{\partial^2 g}{\partial t^2}\\
\frac{\partial^2 f}{\partial x\,\partial y} &= \cos\theta\sin\theta\frac{\partial^2 g}{\partial s^2} + (\cos^2\theta - \sin^2\theta)\frac{\partial^2 g}{\partial s\,\partial t} - \cos\theta\sin\theta\frac{\partial^2 g}{\partial t^2}\\
\frac{\partial^2 f}{\partial y^2} &= \sin^2\theta\frac{\partial^2 g}{\partial s^2} - 2\cos\theta\sin\theta\frac{\partial^2 g}{\partial s\,\partial t} + \cos^2\theta\frac{\partial^2 g}{\partial t^2}\,.
\end{aligned}$$

It follows that

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \Big(\frac{\partial^2 f}{\partial x\,\partial y}\Big)^2 = \frac{\partial^2 g}{\partial s^2}\frac{\partial^2 g}{\partial t^2} - \Big(\frac{\partial^2 g}{\partial s\,\partial t}\Big)^2\,.$$

By a suitable rotation of axes in the xy-plane, we can arrange for the y-axis to be the line of intersection of the xy- and uv-planes. That means that if $x = 0 = z$, then $w = 0$ also; in short, $a_{32} = 0$. We do not need to make much use of this reduction, but it does enable us to dispose of a degenerate case, as we shall do below.

Because O is an orthogonal matrix, its inverse O^{-1} is simply its transpose O^t, that is,

$$O^{-1} = O^t = \begin{pmatrix} a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33}\end{pmatrix}\,,$$

so that we have the set of inverse relations equivalent to those given above:

$$\begin{aligned}
x &= a_{11}u + a_{21}v + a_{31}w\,,\\
y &= a_{12}u + a_{22}v + a_{32}w\,,\\
z &= a_{13}u + a_{23}v + a_{33}w\,.
\end{aligned}$$

The equation

$$O^tO = O\,O^t = I = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\,.$$

can be written as a set of nine equations:

$$\begin{aligned}
a_{11}^2 + a_{12}^2 + a_{13}^2 &= 1\,,\\
a_{21}^2 + a_{22}^2 + a_{23}^2 &= 1\,,\\
a_{31}^2 + a_{32}^2 + a_{33}^2 &= 1\,,\\
a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} &= 0\,,\\
a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} &= 0\,,\\
a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= 0\,,\\
a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= 0\,,\\
a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} &= 0\,,\\
a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= 0\,.
\end{aligned}$$

Only six of these equations are independent, however. An orthogonal 3×3 matrix has three degrees of freedom. These equations imply that $\det(O) = \pm 1$. By interchanging u and v if necessary, we can assume that O is a rotation matrix, that is, its determinant is $+1$. We have no need to do so at the moment, however.

By the chain rule, we find that, on the surface $z = f(x, y)$, we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= a_{11} + a_{13}\frac{\partial f}{\partial x}\,,\\
\frac{\partial u}{\partial y} &= a_{12} + a_{13}\frac{\partial f}{\partial y}\,,\\
\frac{\partial v}{\partial x} &= a_{21} + a_{23}\frac{\partial f}{\partial x}\,,\\
\frac{\partial v}{\partial y} &= a_{22} + a_{23}\frac{\partial f}{\partial y}\,.
\end{aligned}$$

By interchanging u with x, v with y, $f(x, y)$ with $h(u, v)$, and reversing the subscripts, we get the equivalent set of relations

$$\begin{aligned}
\frac{\partial x}{\partial u} &= a_{11} + a_{31}\frac{\partial h}{\partial u}\,,\\
\frac{\partial x}{\partial v} &= a_{21} + a_{31}\frac{\partial h}{\partial v}\,,\\
\frac{\partial y}{\partial u} &= a_{12} + a_{32}\frac{\partial h}{\partial u}\,,\\
\frac{\partial y}{\partial v} &= a_{22} + a_{32}\frac{\partial h}{\partial v}\,.
\end{aligned}$$

The degenerate case just mentioned occurs when $a_{33} = 0$. In that case, given that $a_{32} = 0$, it follows that $a_{31} = \pm 1$, and consequently $a_{11} = 0 = a_{21}$. It is then easy to see that there is an angle θ such that

$$O = \begin{pmatrix} 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \\ \pm 1 & 0 & 0 \end{pmatrix}.$$

Now the equation of the surface is $z = f(x, y)$ in one system of coordinates and $w = h(u, v)$ in the other, where

$$\begin{aligned}
u &= (\cos\theta)y + (\sin\theta)z\,,\\
v &= -(\sin\theta)y + (\cos\theta)z\,,\\
w &= \pm x\,,\\
x &= \pm w\,,\\
y &= (\cos\theta)u - (\sin\theta)v\,,\\
z &= (\sin\theta)u + (\cos\theta)v\,.
\end{aligned}$$

The relation $z = f(x, y)$ is equivalent to $w = h(u, v)$, given these six relations, and that means that we can write an identity in u and v, namely

$$f\big(\pm h(u, v), (\cos\theta)u - (\sin\theta)v\big) = (\sin\theta)u + (\cos\theta)v\,.$$

Differentiating this relation, we get two further identities:

$$\begin{aligned}
\pm\frac{\partial f}{\partial x}\frac{\partial h}{\partial u} + \frac{\partial f}{\partial y}(\cos\theta) &= \sin\theta\,,\\
\pm\frac{\partial f}{\partial x}\frac{\partial h}{\partial v} - \frac{\partial f}{\partial y}(\sin\theta) &= \cos\theta\,.
\end{aligned}$$

But these equations cannot hold at $(0, 0, 0)$, since the two partial derivatives of f vanish at that point. That is, these equations imply that $\sin\theta = 0$ and $\cos\theta = 0$, which is impossible. Therefore we cannot have $a_{33} = 0$ when we choose coordinates so that $a_{32} = 0$, and, as shown above, we can always choose coordinates so that $a_{32} = 0$ without making any change in the expression for curvature in terms of the function f. We are not saying that $a_{33} \neq 0$ in general, only that we could choose coordinates so that this inequality holds, and therefore we shall assume that it does hold from now on.

Having disposed of this degenerate case, we will not make any further use of the assumption that $a_{32} = 0$, but simply assume instead that $a_{33} \neq 0$.

Again by the chain rule, we find, since $f(x, y) = a_{13}u + a_{23}v + a_{33}h(u, v)$, that

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\,,\\
&= \Big(a_{13} + a_{33}\frac{\partial h}{\partial u}\Big)\Big(a_{11} + a_{13}\frac{\partial f}{\partial x}\Big) + \Big(a_{23} + a_{33}\frac{\partial h}{\partial v}\Big)\Big(a_{21} + a_{23}\frac{\partial f}{\partial x}\Big)\,,\\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\,,\\
&= \Big(a_{13} + a_{33}\frac{\partial h}{\partial u}\Big)\Big(a_{12} + a_{13}\frac{\partial f}{\partial y}\Big) + \Big(a_{23} + a_{33}\frac{\partial h}{\partial v}\Big)\Big(a_{22} + a_{23}\frac{\partial f}{\partial y}\Big)\,.
\end{aligned}$$

We rewrite these expressions as

$$\begin{aligned}
\Delta\frac{\partial f}{\partial x} &= -a_{31} + a_{11}\frac{\partial h}{\partial u} + a_{21}\frac{\partial h}{\partial v}\,,\\
\Delta\frac{\partial f}{\partial y} &= -a_{32} + a_{12}\frac{\partial h}{\partial u} + a_{22}\frac{\partial h}{\partial v}\,,
\end{aligned}$$

where

$$\Delta = a_{33} - \Big(a_{13}\frac{\partial h}{\partial u} + a_{23}\frac{\partial h}{\partial v}\Big)\,.$$

To derive the equations for the partial derivatives of f, it is necessary to cancel a_{33} after applying the appropriate equations restricting the entries of the matrix O. That is why we needed to exclude the case when $a_{33} = 0$.

We need these general expressions, since we have to differentiate down to second order to get the curvature. Once we have gotten the second derivatives that we need, however, we are going to evaluate them at $(0, 0)$, and since the partial derivatives of f vanish at that point, the first four of these equations simplify to

$$\begin{aligned}
\frac{\partial u}{\partial x} &= a_{11}\,, \\
\frac{\partial u}{\partial y} &= a_{12}\,, \\
\frac{\partial v}{\partial x} &= a_{21}\,, \\
\frac{\partial v}{\partial y} &= a_{22}\,.
\end{aligned}$$

Since $a_{33} \neq 0$, it follows that, $\Delta \neq 0$ at $(0, 0)$. For, if $\Delta = 0$ at this point, we have the three equations

$$\begin{aligned}
0 &= a_{11}\frac{\partial h}{\partial u} + a_{21}\frac{\partial h}{\partial v} - a_{31}\,, \\
0 &= a_{12}\frac{\partial h}{\partial u} + a_{22}\frac{\partial h}{\partial v} - a_{32}\,, \\
0 &= a_{13}\frac{\partial h}{\partial u} + a_{23}\frac{\partial h}{\partial v} - a_{33}\,,
\end{aligned}$$

which give a non-zero solution, namely

$$X = \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \\ -1 \end{pmatrix}\,,$$

to the homogeneous linear matrix equation

$$O^t X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\,,$$

and that is impossible, since O^t is an invertible matrix. (Actually, as we shall compute below, $\Delta = 1/a_{33}$ at the point $(0, 0)$.)

At the point $(0, 0, 0)$, the equations $\partial f/\partial x = 0$, $\partial f/\partial y = 0$, yield

$$\begin{aligned}
a_{31} &= a_{11}\frac{\partial h}{\partial u} + a_{21}\frac{\partial h}{\partial v}\,, \\
a_{32} &= a_{12}\frac{\partial h}{\partial u} + a_{22}\frac{\partial h}{\partial v}\,.
\end{aligned}$$

Solving these equations for the two partial derivatives of h, we find

$$\begin{aligned}
\frac{\partial h}{\partial u} &= \frac{a_{31}a_{22} - a_{21}a_{32}}{a_{11}a_{22} - a_{12}a_{21}}\,, \\
\frac{\partial h}{\partial v} &= \frac{a_{11}a_{32} - a_{12}a_{31}}{a_{11}a_{22} - a_{12}a_{21}}\,.
\end{aligned}$$

From these equations, we derive the interesting and useful relation

$$
\begin{aligned}
(a_{11}a_{22} - a_{12}a_{21})^2 & \left(1 + \left(\frac{\partial h}{\partial u}\right)^2 + \left(\frac{\partial h}{\partial v}\right)^2\right) = \\
&= (a_{11}a_{22} - a_{12}a_{21})^2 + (a_{31}a_{22} - a_{21}a_{32})^2 + (a_{11}a_{32} - a_{12}a_{31})^2 , \\
&= a_{11}^2(a_{22}^2 + a_{32}^2) + a_{21}^2(a_{12}^2 + a_{32}^2) + a_{31}^2(a_{12}^2 + a_{22}^2) \\
&\quad - 2(a_{11}a_{21}a_{12}a_{22} + a_{21}a_{31}a_{22}a_{32} + a_{11}a_{31}a_{12}a_{32}) \\
&= a_{11}^2(1 - a_{12}^2) + a_{21}^2(1 - a_{22}^2) + a_{31}^2(1 - a_{32}^2) \\
&\quad - 2(a_{11}a_{21}a_{12}a_{22} + a_{21}a_{31}a_{22}a_{32} + a_{11}a_{31}a_{12}a_{32}) \\
&= 1 - (a_{11}^2a_{12}^2 + a_{21}^2a_{22}^2 + a_{31}^2a_{32}^2 \\
&\quad - 2(a_{11}a_{21}a_{12}a_{22} + a_{21}a_{31}a_{22}a_{32} + a_{11}a_{31}a_{12}a_{32}) \\
&= 1 - (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})^2 = 1 .
\end{aligned}
$$

That is, we find

$$
1 + \left(\frac{\partial h}{\partial u}\right)^2 + \left(\frac{\partial h}{\partial v}\right)^2 = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})^2} .
$$

Now, if we repeat some of this argument, basing it on the relation $h(u,v) = a_{31}x + a_{32}y + a_{33}f(x,y)$, we find that

$$
\begin{aligned}
\frac{\partial h}{\partial u} &= \frac{\partial h}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial u} , \\
&= \left(a_{31} + a_{33}\frac{\partial f}{\partial x}\right)\left(a_{11} + a_{31}\frac{\partial h}{\partial u}\right) + \left(a_{32} + a_{33}\frac{\partial f}{\partial y}\right)\left(a_{12} + a_{32}\frac{\partial h}{\partial u}\right) , \\
\frac{\partial h}{\partial v} &= \frac{\partial h}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial v} , \\
&= \left(a_{31} + a_{33}\frac{\partial f}{\partial x}\right)\left(a_{21} + a_{31}\frac{\partial h}{\partial v}\right) + \left(a_{32} + a_{33}\frac{\partial f}{\partial y}\right)\left(a_{22} + a_{32}\frac{\partial h}{\partial v}\right) .
\end{aligned}
$$

We need these relations only at $(0,0,0)$, where they simplify to

$$
\begin{aligned}
\frac{\partial h}{\partial u} &= a_{31}\left(a_{11} + a_{31}\frac{\partial h}{\partial u}\right) + a_{32}\left(a_{12} + a_{32}\frac{\partial h}{\partial u}\right) , \\
\frac{\partial h}{\partial v} &= a_{31}\left(a_{21} + a_{31}\frac{\partial h}{\partial v}\right) + a_{32}\left(a_{22} + a_{32}\frac{\partial h}{\partial v}\right) .
\end{aligned}
$$

That is,

$$
\begin{aligned}
a_{33}^2\frac{\partial h}{\partial u} &= a_{11}a_{31} + a_{12}a_{32} = -a_{13}a_{33} , \\
a_{33}^2\frac{\partial h}{\partial v} &= a_{31}a_{21} + a_{32}a_{22} = -a_{33}a_{23} .
\end{aligned}
$$

It follows that

$$\begin{aligned}\frac{\partial h}{\partial u} &= -\frac{a_{13}}{a_{33}}\,,\\ \frac{\partial h}{\partial v} &= -\frac{a_{23}}{a_{33}}\,.\end{aligned}$$

As a result, we find that, at the origin,

$$1+\Bigl(\frac{\partial h}{\partial u}\Bigr)^2+\Bigl(\frac{\partial h}{\partial v}\Bigr)^2=\frac{1}{a_{33}^2}\,.$$

The last fact we need before computing the curvature in terms of $h(u,v)$ is the relation

$$\begin{aligned}\Delta(0,0) &= a_{33}-\Bigl(a_{13}\frac{\partial h}{\partial u}+a_{23}\frac{\partial h}{\partial v}\Bigr)\\ &= a_{33}+\frac{a_{13}^2}{a_{33}}+\frac{a_{23}^2}{a_{33}}=\frac{1}{a_{33}}\,.\end{aligned}$$

so that, at the origin,

$$\Delta^{-2}=a_{33}^2=1+\Bigl(\frac{\partial h}{\partial u}\Bigr)^2+\Bigl(\frac{\partial h}{\partial v}\Bigr)^2\,.$$

Since we can now assume that $\Delta \neq 0$, let us introduce the abbreviations

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{N_1}{\Delta}\,,\\ \frac{\partial f}{\partial y} &= \frac{N_2}{\Delta}\,,\end{aligned}$$

where

$$\begin{aligned}N_1 &= -a_{13}+a_{11}\frac{\partial h}{\partial u}+a_{12}\frac{\partial h}{\partial v}\,,\\ N_2 &= -a_{23}+a_{21}\frac{\partial h}{\partial u}+a_{22}\frac{\partial h}{\partial v}\,.\end{aligned}$$

We need to compute the derivatives of these fractions and evaluate them at $(0,0)$. We get a break in doing so, since $N_1 = 0 = N_2$ at that point. Thus, the equations we get will be

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{1}{\Delta}\frac{\partial N_1}{\partial x}\,,\\ \frac{\partial^2 f}{\partial x\,\partial y} &= \frac{1}{\Delta}\frac{\partial N_1}{\partial y}=\frac{1}{\Delta}\frac{\partial N_2}{\partial x}\,,\\ \frac{\partial^2 f}{\partial y^2} &= \frac{1}{\Delta}\frac{\partial N_2}{\partial y}\,.\end{aligned}$$

The derivation is only a little bit tedious. We find that, at the origin,

$$\begin{aligned}
\frac{\partial N_1}{\partial x} &= a_{11}^2\frac{\partial^2 h}{\partial u^2} + 2a_{11}a_{21}\frac{\partial^2 h}{\partial u\,\partial v} + a_{21}^2\frac{\partial^2 h}{\partial v^2}\,,\\
\frac{\partial N_1}{\partial y} &= a_{11}a_{12}\frac{\partial^2 h}{\partial u^2} + (a_{11}a_{22}+a_{12}a_{21})\frac{\partial^2 h}{\partial u\,\partial v} + a_{21}a_{22}\frac{\partial^2 h}{\partial v^2}\,,\\
\frac{\partial N_2}{\partial y} &= a_{12}^2\frac{\partial^2 h}{\partial u^2} + 2a_{12}a_{22}\frac{\partial^2 h}{\partial u\,\partial v} + a_{22}^2\frac{\partial^2 h}{\partial v^2}\,.
\end{aligned}$$

Now, we know that the curvature at $(0,0)$ is

$$\begin{aligned}
\kappa(0,0) &= \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \Big(\frac{\partial^2 f}{\partial x\,\partial y}\Big)^2\\
&= \Delta^{-2}\bigg(\frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial y} - \Big(\frac{\partial N_1}{\partial y}\Big)^2\bigg).
\end{aligned}$$

Multiplying the trinomial expressions for the partial derivatives of N_1 and N_2 and using the relations that make O an orthogonal matrix, we find at last that, as asserted,

$$\begin{aligned}
\kappa &= a_{33}^2(a_{11}a_{22}-a_{21}a_{12})^2\bigg(\frac{\partial^2 h}{\partial u^2}\frac{\partial^2 h}{\partial v^2} - \Big(\frac{\partial^2 h}{\partial u\,\partial v}\Big)^2\bigg)\\
&= \frac{\frac{\partial^2 h}{\partial u^2}\frac{\partial^2 h}{\partial v^2} - (\frac{\partial^2 h}{\partial u\,\partial v}{}^2}{\big(1+(\frac{\partial h}{\partial u})^2+(\frac{\partial h}{\partial v})^2\big)^2}\,.
\end{aligned}$$

PROBLEM 5.3. Compute the Gaussian curvature of the upper pseudo-hemisphere (see Appendix 1) using parametrization by a length $u>0$ representing latitude and an angle v representing longitude, as follows:

$$\boldsymbol{r}(u,v) = \bigg(a\mathrm{sech}\left(\frac{u}{a}\right)\cos(v), a\mathrm{sech}\left(\frac{u}{a}\right)\sin(v), u - a\tanh\left(\frac{u}{a}\right)\bigg).$$

Solution: It is highly recommended that *Mathematica* be used to carry out the computations, unless the reader is very familiar with trigonometric and hyperbolic identities and in addition very patient and accurate. We find that

$$\boldsymbol{n}(u,v) = \bigg(-\tanh\left(\frac{u}{a}\right)\cos(v) - \tanh\left(\frac{u}{a}\right)\sin(v), -\mathrm{sech}\left(\frac{u}{a}\right)\bigg).$$

The constants from the two fundamental forms are computed as

$$\begin{aligned}
E &= \tanh^2\left(\frac{u}{a}\right),\\
F &= 0\,,\\
G &= a^2\mathrm{sech}^2\left(\frac{u}{a}\right),\\
D &= \frac{1}{a^2}\mathrm{sech}^2\left(\frac{u}{a}\right),\\
D' &= 0\,,\\
D'' &= \tanh^2\left(\frac{u}{a}\right).
\end{aligned}$$

From these formulas, we readily compute that

$$\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \cdot \frac{\partial \boldsymbol{n}}{\partial u} \times \frac{\partial \boldsymbol{n}}{\partial v} = -\operatorname{sech}^2\Bigl(\frac{u}{a}\Bigr) \tanh^2\Bigl(\frac{u}{a}\Bigr)\,,$$

so that the curvature of this surface at every point is negative. The curvature is

$$\kappa(u,v) = \frac{-1}{a^2}\,.$$

This justifies calling the surface a pseudo-sphere. In analogy with the sphere, it has constant curvature, but that curvature is negative rather than positive.

PROBLEM 5.4. Show how to express the curvature of a surface $z = f(x,y)$ using only the metric coefficients $E = 1 + \bigl(\frac{\partial f}{\partial x}\bigr)^2$, $F = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$, and $G = 1 + \bigl(\frac{\partial f}{\partial y}\bigr)^2$.

Solution: Here is a *Mathematica* notebook that does the labor.

Mathematica NOTEBOOK 1.

```
κ[x_, y_] := (Dot[D[r[x,y],x,2],D[r[x,y], x]]
Dot[D[r[x,y], y,2], D[r[x,y],y]] - Dot[D[D[r[x,y], x],y],D[r[x,y],x]]
Dot[D[D[r[x,y],x],y],D[r[x,y],y]])/(Dot[D[r[x,y],x],D[r[x,y],y]]
(Dot[D[r[x,y],x],D[r[x,y],x]]Dot[D[r[x,y],y],D[r[x,y],y]]
- Dot[D[r[x,y],x],D[r[x,y],y]]^2)^2);
```

Here are three sample surfaces for testing it. *Mathematica* responds very quickly:

A sphere of radius R:

```
r[x_, y_] := {x, y, Sqrt[R^2 - x^2 - y^2]}; FullSimplify[κ[x,y]]
```

The hyperbolic paraboloid:

```
r[x_, y_] := {x, y, (x^2 - y^2)/a]}; FullSimplify[κ[x,y]]
```

And a cubic surface:

```
r[x_, y_] := {x, y, x^3 - y^2]}; FullSimplify[κ[x,y]]
```

REMARK 5.1. This expression shows that when we use these "privileged coordinates" of Euler, we actually do not need *complete* information about the second-order partial derivatives of $\boldsymbol{r}$ in order to compute the curvature; all we really need is the values of their dot products with the two tangent vectors that are the first-order derivatives. What we really need are the projections of these second derivatives on the tangent plane. But, as we noted in the case of the curvature of a one-dimensional manifold in $\mathbb{R}^n$, it is the component of the second derivative *perpendicular* to the tangent line that gives the curvature. We can thus count ourselves very lucky to be able to get the curvature from information available in the tangent plane, and we shouldn't expect to be this lucky in general.

The trouble with the expression for the curvature just given is that the derivatives are taken with respect to x and y rather than with respect to u and v. When we try to translate it into (u,v)-coordinates, coefficients appear, each of which has to be computed as the first or second derivative of some component of the original mapping $\boldsymbol{r}(u,v)$. Thus, we once again need to know more than just the three metric coefficients E, F, and G.

PROBLEM 5.5. Show that $\mathrm{Rie}\bigl((1,0),(0,1),(1,0),(0,1)\bigr)$ is the right-hand side of Gauss's expression for the curvature divided by $4(EG - F^2)$, that is, Eq. (68).

Solution: It would be foolish not to take advantage of *Mathematica* here. The following quick-and-dirty sequence of commands gets the result:

```
p = {u, v}; gsub = {{e[u, v], f[u, v]}, {f[u, v], g[u, v]}};
gsup = Inverse[gsub]; Γ = FullSimplify[Table[(1/2) Sum[gsup[[i,q]]
(D[gsub[[j, q]], p[[k]]] + D[gsub[[q, k]], p[[j]]]
- D[gsub[[j, k]], p[[q]]]),{q, 1, Length[p]}], {i, 1, Length[p]},
{j, 1, Length[p]}, {k, 1, Length[p]}]];
Riem = FullSimplify[Table[D[Γ[[i, j, l]], p[[k]]]
- D[Γ[[i, j, k]], p[[l]]] + Sum[Γ[[n, j, l]] Γ[[i, k, n]]
- Γ[[n, j, k]] Γ[[i, l, n]], {n, 1, Length[p]}], {i, 1, Length[p]},
{j, 1, Length[p]}, {k, 1, Length[p]}, {l, 1, Length[p]}]];
CovRiem = FullSimplify[Table[Sum[gsub[[i, m]] Riem[[m, j, k, l]],
{m, 1, 2}], {i, 1, 2}, {j, 1, 2}, {k, 1, 2}, {l, 1, 2}]];
vec = {{1, 0}, {0, 1}, {1, 0}, {0, 1}};
FullSimplify[Sum[CovRiem[[i, j, k, l]]
vec[[1, i]] vec[[2, j]] vec[[3, k]] vec[[4,l]],
{i, 1, 2}, {j, 1, 2}, {k, 1, 2}, {l, 1, 2}]]
```

The output, though hard to read, is exactly what Gauss wrote, except that we have used lower-case letters, since *Mathematica* will not allow the symbol E to be used as a variable.

PROBLEM 5.6. Apply Gauss's formula for the curvature to the case of a parametrization $(u, v) \mapsto \bigl(u, v, f(u, v)\bigr)$, and show that the second-order derivatives of E, F, and G that occur in formula (68) are such that the third-order derivatives of $\boldsymbol{r}$ (which involve the derivatives of the Christoffel symbols), cancel one another out. Specifically, show that

$$\frac{\partial^2 E}{\partial v^2} - 2\frac{\partial^2 F}{\partial u\,\partial v} + \frac{\partial^2 G}{\partial u^2} = 4\Bigl(\frac{\partial^2 f}{\partial u\,\partial v}\Bigr)^2 - 2\frac{\partial^2 f}{\partial u^2}\,\frac{\partial^2 f}{\partial v^2}\,.$$

Solution: For this special case, we have

$$\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial u} &= \Bigl(1, 0, \frac{\partial f}{\partial u}\Bigr)\,,\\
\frac{\partial \boldsymbol{r}}{\partial v} &= \Bigl(0, 1, \frac{\partial f}{\partial v}\Bigr)\,,\\
E &= 1 + \Bigl(\frac{\partial f}{\partial u}\Bigr)^2\,,\\
F &= \frac{\partial f}{\partial u}\,\frac{\partial f}{\partial v}\,,\\
G &= 1 + \Bigl(\frac{\partial f}{\partial v}\Bigr)^2\,.
\end{aligned}$$

The required equation follows from this by trivial computation. In particular, the third-order derivatives of f cancel out, as asserted.

PROBLEM 5.7. Show that $\mathrm{Rie}\,(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}) \equiv -\mathrm{Rie}\,(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{z})$.

Solution: We recall that the coefficients of Rie are R_{ijkl}, given by

$$g_{mi}\Bigl(\frac{\partial \Gamma^m_{lj}}{\partial x^k} - \frac{\partial \Gamma^m_{kj}}{\partial x^l} + \bigl(\Gamma^m_{kp}\Gamma^p_{lj} - \Gamma^m_{lp}\Gamma^p_{kj}\bigr)\Bigr)\,.$$

We need to show that $R_{ijkl} + R_{jikl} = 0$. Our method of proof consists of the following steps:

(1) Find an expression for $\partial\Gamma^m_{lj}/\partial x^k$ in terms of the metric and Christoffel coefficients.
(2) Multiply the result of Step 1 by g_{im}.
(3) Reverse the indices k and l, and subtract the result from the result of Step 2.
(4) Reverse the indices i and j in the result of Step 3 and add the result to the result of Step 3.
(5) Reverse the indices i and j again and add the expression $g_{im}\bigl(\Gamma^m_{kp}\Gamma^p_{lj} - \Gamma^m_{lp}\Gamma^p_{kj}\bigr)$ to the output of Step 3.
(6) Subtract the result of Step 5 from the result of Step 4. The difference is $R_{ijkl} + R_{jikl}$, and it turns out to be zero.

Since the algebra involved is rather tedious, we merely sketch the details. The rest is just a matter of working carefully. Step 1 is the only one that requires some care in handling. By definition of the Christoffel coefficients, we have

$$\frac{\partial\Gamma^m_{lj}}{\partial x^k} = \frac{1}{2}\frac{\partial g^{mp}}{\partial x^k}\Bigl(\frac{\partial g_{pl}}{\partial x^j} + \frac{\partial g_{pj}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^p}\Bigr) + \frac{1}{2}g^{mp}\Bigl(\frac{\partial^2 g_{pl}}{\partial x^k\partial x^j} + \frac{\partial^2 g_{pj}}{\partial x^k\partial x^l} - \frac{\partial^2 g_{lj}}{\partial x^k\partial x^p}\Bigr)\,.$$

Now, without any further transformations, the second term here (containing all the second-order partial derivatives) yields zero as output when it is subjected to the transformations in Steps 2, 3, and 4. Thus we may ignore it and concentrate on the first term, which contains only first-order derivatives of the metric coefficients. We deal with that expression using the formulas proved in Problems 4.12–4.13:

$$\begin{aligned}\frac{\partial g_{ij}}{\partial x^k} &= \Gamma^l_{ik}g_{lj} + \Gamma^l_{jk}g_{li}\,,\\ \frac{\partial g^{ij}}{\partial x^k} &= -\bigl(\Gamma^j_{mk}g^{im} + \Gamma^i_{mk}g^{jm}\bigr)\,.\end{aligned}$$

With only a small amount of trouble, we find that the first term equals

$$-\,\bigl(\Gamma^m_{pk}\Gamma^p_{lj} + \Gamma^p_{qk}\Gamma^r_{lj}g^{mq}g_{rp}\bigr)\,.$$

The ambitious student who cares to differentiate the two expressions and get similar expressions for the second-order partial derivatives of the metric coefficients will find—after perhaps half an hour of tedious algebra—that the sum of the terms involving second-order partial derivatives is equal to

$$\frac{\partial\Gamma^m_{lj}}{\partial x^k} + \Gamma^m_{pk}\Gamma^p_{lj} + \Gamma^p_{qk}\Gamma^r_{lj}g^{mq}g_{rp}\,,$$

and thus we have a check on the validity of our formulas and the accuracy of our work. As noted, however, it is not difficult to show that the transformations in this process annihilate the second term, so that we have no need to expose ourselves to the tedium.

At this point, the proof consists of merely routine application of algebra and the fundamental relation $g^{ab}g_{ac} = \delta^b_c$.

CHAPTER 6

Concepts of Curvature, 1850–1950

PROBLEM 6.1. A vector $\boldsymbol{a} = (a^1, a^2, a^3)$ in $\mathbb{R}^3$ can be naturally associated with a skew-symmetric 3×3 matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & a^3 & a^2 \\ -a^3 & 0 & a^1 \\ -a^2 & -a^1 & 0 \end{pmatrix} .$$

Show that, if $\boldsymbol{b} = (b^1, b^2, b^3)$ is associated in this way with the matrix $\boldsymbol{B}$, then the cross product $\boldsymbol{a} \times \boldsymbol{b}$ is associated with $[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{A}\,\boldsymbol{B} - \boldsymbol{B}\,\boldsymbol{A}$. (Replacing an associative product with its commutator, that is, replacing AB with $[A, B]$ is a standard way of turning an associative algebra into a Lie algebra. If the associative algebra happens to be commutative, of course, the Lie algebra is trivial, since the Lie products are all equal to zero.)

Solution: This is a routine computation. For example, the element in row 1, column 2 of $\boldsymbol{A}\,\boldsymbol{B}$ is $-a^2b^1$, and so obviously the element in row 1, column 2 of $\boldsymbol{B}\,\boldsymbol{A}$ is $-b^2a^1$. Hence the element in that place in $[\boldsymbol{A}, \boldsymbol{B}]$ is $a^1b^2 - a^2b^1$, which is the third component of the cross product $\boldsymbol{a} \times \boldsymbol{b}$. The other entries are handled similarly.

PROBLEM 6.2. Consider a general surface in $\mathbb{R}^3$ parameterized by u and v, that is, $(u, v) \to \boldsymbol{r}(u, v)$, and a curve $\boldsymbol{\gamma}(s)$ on that surface:

$$\boldsymbol{\gamma}(s) = \boldsymbol{r}\big(u(s), v(s)\big) .$$

For a fixed parameter value $s = s_0$, let $u_0 = u(s_0)$, $v_0 = v(s_0)$. Let $\boldsymbol{v}(u_0, v_0)$ be a vector at the point $P_0 = \boldsymbol{r}(u_0, v_0)$ on the surface:

$$\boldsymbol{v}(u_0, v_0) = a(u_0, v_0)\frac{\partial \boldsymbol{r}}{\partial u} + b(u_0, v_0)\frac{\partial \boldsymbol{r}}{\partial v} .$$

Show that, if $\boldsymbol{v}\big(u(s), v(s)\big)$ is the parallel transport of $\boldsymbol{v}\big(u_0, v_0)\big)$ from the point P_0 along the curve, that is

$$\boldsymbol{v}\big(u(s), v(s)\big) = a\big(u(s), v(s)\big)\frac{\partial \boldsymbol{r}}{\partial u} + b\big(u(s), v(s)\big)\frac{\partial \boldsymbol{r}}{\partial v} ,$$

then the squared length of the vector $\boldsymbol{v}\big(u(s), v(s)\big)$, which is

$$\Big(a\big(u(s), v(s)\big)\Big)^2 E\big(u(s), v(s)\big) + 2a\big(u(s), v(s)\big)b\big(u(s), v(s)\big)F\big(u(s), v(s)\big) \\ + \Big(b\big(u(s), v(s)\big)\Big)^2 G\big(u(s)v(s)\big) ,$$

is constant. (That is, the derivative of this expression with respect to s is zero.)

Solution: Although the algebra is slightly tedious, one can give the following generic formula for the covariant derivative:

$$\begin{aligned}\nabla_{\boldsymbol{\gamma}'(s)} = \Bigl(u'\,\frac{\partial a}{\partial u} + v'\,\frac{\partial a}{\partial v} + au'\Gamma^1_{11} + bu'\Gamma^1_{12} + av'\Gamma^1_{12} + bv'\Gamma^1_{22}\Bigr)\frac{\partial \boldsymbol{r}}{\partial u}\\ + \Bigl(u'\,\frac{\partial b}{\partial u} + v'\,\frac{\partial b}{\partial v} + au'\Gamma^2_{11} + bu'\Gamma^2_{12} + av'\Gamma^2_{12} + bv'\Gamma^2_{22}\Bigr)\frac{\partial \boldsymbol{r}}{\partial v}\,.\end{aligned}$$

Given that formula, it is once again tedious, but not at all difficult, to verify the identity

$$\frac{d}{ds}\Bigl(a^2E + 2abF + b^2G\Bigr) = 2\nabla_{\boldsymbol{\gamma}'(s)}(a,b)\cdot\Bigl(a\frac{\partial \boldsymbol{r}}{\partial u} + b\frac{\partial \boldsymbol{r}}{\partial v}\Bigr)\,.$$

(There are 24 terms on each side of this equation, and they match up perfectly in pairs.) Thus, the vanishing of the covariant derivative along the path ensures that the length of the transported vector remains constant.

PROBLEM 6.3. Finding geodesics is a task that can take bizarre twists. In Example 6.3, we found the complete family of geodesics on the pseudo-sphere through a given point, despite the fact that this surface has a complicated equation involving transcendental functions. To be sure, when cylindrical coordinates are used, z is independent of θ, and that makes the task much easier. Even so, compared to that example, one would expect it to be utterly trivial to find the geodesics on a surface as simple (algebraically) as the hyperbolic paraboloid whose equation is $z = xy/c$. In the natural parametrization $\boldsymbol{r}(x,y) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + (xy/c)\,\boldsymbol{k}$, we have $g(0,0) = I$, and so all we need to do is find the geodesic $\boldsymbol{\gamma}(t)$ with $\boldsymbol{\gamma}(0) = \mathbf{0}$ whose tangent at $\mathbf{0}$ when arc length is the parameter is given by

$$\boldsymbol{\gamma}'(0) = \frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{j}\,.$$

for each pair x, y with $x^2 + y^2 = 1$. In polar coordinates, we need

$$\begin{aligned}\boldsymbol{\gamma}(0) &= \mathbf{0}\,,\\ \boldsymbol{\gamma}'(0) &= \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}\,.\end{aligned}$$

Show that the Euler equations for this problem imply the equations

$$\begin{aligned}x' &= v\,,\\ y' &= w\,,\\ xv' - yw' &= 0\,,\\ v^2 + w^2 + (xw+yv)^2/c^2 &= 1\,.\end{aligned}$$

(Attempt to solve these equations at your own risk!!)

Solution: Using the parametrization $\boldsymbol{r}(x,y) = (x, y, xy/c)$, we easily get $E = 1 + y^2/c^2$, $F = xy/c^2$, $G = 1 + x^2/c^2$. Hence

$$ds^2 = \Bigl(1 + \frac{y^2}{c^2}\Bigr)\,dx^2 + 2\frac{xy}{c^2}\,dx\,dy + \Bigl(1 + \frac{x^2}{c^2}\Bigr)\,dy^2\,.$$

Assuming arc length as the parameter, we find the following Euler equations, after multiplying by $c^2/2$:

$$\begin{aligned}\frac{d}{ds}\bigl((c^2+y^2)x' + (xy)y'\bigr) &= y'(x'y + xy') = y'(xy)'\\ \frac{d}{ds}\bigl((c^2+x^2)y' + (xy)x'\bigr) &= x'(x'y + xy') = x'(xy)'\,.\end{aligned}$$

Thus

$$\begin{aligned} c^2x'' + \frac{d}{ds}(y(xy)') &= y'(xy)' \\ c^2y'' + \frac{d}{ds}(x(xy)') &= x'(xy)' . \end{aligned}$$

These equations immediately simplify to

$$\begin{aligned} c^2x'' + y(xy)'' &= 0 , \\ c^2y'' + x(xy)'' &= 0 . \end{aligned}$$

We then see easily that $xx'' - yy'' = 0$. Thus, if we define $v = x'$, $w = y'$, we get the first three equations immediately. And since $xw + yv = (xy)'$, it remains only to show that

$$(x')^2 + (y')^2 + \bigl((xy)'\bigr)^2/c^2 = 1 .$$

But that is merely the equation $(x')^2+(y')^2+(z')^2 = 1$, that is, $ds^2 = dx^2+dy^2+dz^2$, which is precisely the metric on $\mathbb{R}^3$.

Problem 6.4. Show that Eq. (80) becomes $\theta = \theta_1$ when $a = c$. That is, it is a geodesic that is a meridian of longitude. For $a < c$, show that Eq. (80) implies Eq. (78) with $r_0 = \sqrt{c^2 - a^2}$ and $\theta_0 = \theta_1 + c\sqrt{a^2 - c^2 + r_1^2}/(r_1\sqrt{c^2 - a^2})$.

Solution: The first assertion is trivial, since all terms except the term θ_1 on the right-hand side of Eq. (80) contain the factor $\sqrt{c^2 - a^2}$.

As for the second assertion, it is a slightly tedious computation when θ_1 is replaced by the expression in terms of θ_0. One has only to add the first two terms, cancel, then add the third term to the result and cancel again.

Problem 6.5. Prove the formulas

$$[\boldsymbol{u}, \boldsymbol{v}] = \Bigl(u^l\frac{\partial v^i}{\partial x^l} - v^l\frac{\partial u^i}{\partial x^l}\Bigr)\frac{\partial}{\partial x^i} = \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} ,$$

and

$$\nabla_{[\boldsymbol{u},\boldsymbol{v}]} = \nabla_{\nabla_{\boldsymbol{u}}\boldsymbol{v}} - \nabla_{\nabla_{\boldsymbol{v}}\boldsymbol{u}} .$$

Solution: The first of these equalities is a routine computation. For any C^∞-function f, we have

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{v}f) &= \boldsymbol{u}\Bigl(v^i\frac{\partial f}{\partial x^i}\Bigr) \\ &= u^l\frac{\partial}{\partial x^j}\Bigl(v^l\frac{\partial f}{\partial x^l}\Bigr) \\ &= u^l\Bigl(\frac{\partial v^i}{\partial x^l}\frac{\partial f}{\partial x^i} + v^i\frac{\partial^2 f}{\partial x^l\,\partial x^i}\Bigr) . \end{aligned}$$

By symmetry, we get

$$\boldsymbol{v}(\boldsymbol{u}f) = v^l\Bigl(\frac{\partial u^i}{\partial x^l}\frac{\partial f}{\partial x^i} + u^i\frac{\partial^2 f}{\partial x^l\,\partial x^i}\Bigr) .$$

Subtracting this last equation from the one before yields the stated expression for $[\boldsymbol{u}, \boldsymbol{v}]f$, that is,

$$[\boldsymbol{u}, \boldsymbol{v}] = \Bigl(u^l\frac{\partial v^i}{\partial x^l} - v^l\frac{\partial u^i}{\partial x^l}\Bigr)\frac{\partial}{\partial x^i} .$$

Then, since

$$\nabla_{\boldsymbol{u}}\boldsymbol{v} = u^k\Big(\frac{\partial v^i}{\partial x^k} + v^j\Gamma^i_{jk}\Big)\frac{\partial}{\partial x^i}\,,$$

we see that the Christoffel symbols cancel out of the expression $\nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u}$, leaving an expression identical to the expression for $[\boldsymbol{u}, \boldsymbol{v}]$.

We then get

$$\nabla_{[\boldsymbol{u},\boldsymbol{v}]}(\boldsymbol{w}) = \Bigg(\Big(u^l\frac{\partial v^k}{\partial x^l} - v^l\frac{\partial u^k}{\partial x^l}\Big)\Big(\frac{\partial w^i}{\partial x^k} + w^j\Gamma^i_{jk}\Big)\,.$$

Also,

$$\begin{aligned}\nabla_{\nabla_{\boldsymbol{u}}\boldsymbol{v}}(\boldsymbol{w}) &= \Big(u^k\frac{\partial v^i}{\partial x^k} + v^j u^k\Gamma^i_{jk}\Big)\nabla_{\frac{\partial}{\partial x^i}}(\boldsymbol{w})\\ &= \Big(u^k\frac{\partial v^i}{\partial x^k} + u^k v^j\Gamma^i_{jk}\Big)\Big(\frac{\partial w^l}{\partial x^i} + w^k\Gamma^l_{ik}\Big)\frac{\partial}{\partial x^l}\,.\end{aligned}$$

Now when we interchange $\boldsymbol{u}$ and $\boldsymbol{v}$ and subtract the result from this last equation, we find that the first factor becomes precisely $[\boldsymbol{u}, \boldsymbol{v}]^i$, which establishes the result.

PROBLEM 6.6. Our derivation of the Bianchi identity suggests that it is possible to take the Lie derivative not only of a tangent vector but also of a linear mapping M from the tangent space to the set of linear operators on the space, producing in effect a trilinear mapping of triples of tangent vectors. The definition is

$$\big(L(\boldsymbol{u})M\big)(\boldsymbol{v}, \boldsymbol{w}) = L(\boldsymbol{u})\big(M(\boldsymbol{v})\boldsymbol{w}\big) - \big(M(L(\boldsymbol{u})\boldsymbol{v})\boldsymbol{w} - M(\boldsymbol{v})\big(L(\boldsymbol{u})\boldsymbol{w}\big)\,.$$

Show that the Lie derivative of the covariant derivative operator ∇ regarded as the mapping from the tangent space to the space of linear operators on the tangent space given by the mapping $\boldsymbol{u} \mapsto \nabla_{\boldsymbol{u}}$, satisfies

$$\big(L(\boldsymbol{u})\nabla\big)(\boldsymbol{v}, \boldsymbol{w}) = R(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w} + \nabla^2_{\boldsymbol{v},\boldsymbol{w}}\boldsymbol{u}\,.$$

Solution: By definition, $\nabla(\boldsymbol{u}) = \nabla_{\boldsymbol{u}}$, and so

$$\big(L(\boldsymbol{u})\nabla\big)(\boldsymbol{v}, \boldsymbol{w}) = \big[\boldsymbol{u}, \nabla_{\boldsymbol{v}}\boldsymbol{w}\big] - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w} - \nabla_{\boldsymbol{v}}[\boldsymbol{u}, \boldsymbol{w}]$$

Also by definition $\nabla^2_{\boldsymbol{v},\boldsymbol{w}}\boldsymbol{u} = \nabla_{\boldsymbol{v}}\big(\nabla_{\boldsymbol{w}}\boldsymbol{u}\big) - \nabla_{\nabla_{\boldsymbol{v}}\boldsymbol{w}}\boldsymbol{u}$

It was proved in the text that

$$R(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w} = \nabla_{\boldsymbol{u}}\big(\nabla_{\boldsymbol{v}}\boldsymbol{w}\big) - \nabla_{\boldsymbol{v}}\big(\nabla_{\boldsymbol{u}}\boldsymbol{w}\big) - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}\,.$$

Thus, canceling the term $\nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}$ on both sides, we need to show that

$$\big[\boldsymbol{u}, \nabla_{\boldsymbol{v}}\boldsymbol{w}\big] - \nabla_{\boldsymbol{v}}[\boldsymbol{u}, \boldsymbol{w}] = \nabla_{\boldsymbol{u}}\big(\nabla_{\boldsymbol{v}}\boldsymbol{w}\big) - \nabla_{\boldsymbol{v}}\big(\nabla_{\boldsymbol{u}}\boldsymbol{w}\big) + \nabla_{\boldsymbol{v}}\big(\nabla_{\boldsymbol{w}}\boldsymbol{u}\big) - \nabla_{\nabla_{\boldsymbol{v}}\boldsymbol{w}}\boldsymbol{u}\,.$$

By the previous problem,

$$[\boldsymbol{u}, \boldsymbol{v}] = \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u}$$

and

$$\nabla_{[\boldsymbol{u},\boldsymbol{v}]}(\boldsymbol{w}) = \nabla_{\nabla_{\boldsymbol{u}}\boldsymbol{v}}\boldsymbol{w} - \nabla_{\nabla_{\boldsymbol{v}}\boldsymbol{u}}\boldsymbol{w}\,.$$

If we use these formulas to replace the Lie brackets on the left-hand side, we find that the resulting terms match up one-to-one with the terms on the right.

PROBLEM 6.7. Show that

$$\big(L(\boldsymbol{u})L\big)(\boldsymbol{v},\boldsymbol{w}) = \big(L(\boldsymbol{u})\nabla\big)(\boldsymbol{v},\boldsymbol{w}) - \big(L(\boldsymbol{u})\nabla\big)(\boldsymbol{w},\boldsymbol{v})\,.$$

and that

$$\big(L(\boldsymbol{u})\nabla\big)(\boldsymbol{w},\boldsymbol{v}) = \big(L(\boldsymbol{u})\nabla\big)(\boldsymbol{v},\boldsymbol{w})\,.$$

Solution: By the previous problem, the right-hand side is

$$R(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} + \nabla^2_{\boldsymbol{v},\boldsymbol{w}}\boldsymbol{u} - R(\boldsymbol{u},\boldsymbol{w})\boldsymbol{v} - \nabla^2_{\boldsymbol{w},\boldsymbol{v}}\boldsymbol{u}\,.$$

But it was shown in the text that $\nabla^2_{\boldsymbol{v},\boldsymbol{w}}\boldsymbol{u} - \nabla^2_{\boldsymbol{w},\boldsymbol{v}}\boldsymbol{u} = R(\boldsymbol{v},\boldsymbol{w})\boldsymbol{u}$. Hence the right-hand side is

$$R(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} + R(\boldsymbol{w},\boldsymbol{u})\boldsymbol{v} + R(\boldsymbol{v},\boldsymbol{w})\boldsymbol{u}\,,$$

which is zero by the Bianchi identity. The left-hand side, when sorted out, is just

$$\big[\boldsymbol{u},[\boldsymbol{v},\boldsymbol{w}]\big] + \big[\boldsymbol{w},[\boldsymbol{u},\boldsymbol{v}]\big] + \big[\boldsymbol{v},[\boldsymbol{w},\boldsymbol{u}]\big]\,,$$

which is also zero, being precisely the Jacobi identity.

PROBLEM 6.8. Consider a coordinate system (for example, normal coordinates) in which the matrix of metric coefficients is the identity matrix at the point P. Prove that in these coordinates $R^i_{jkl} = -R^j_{ikl}$.

Solution: By differentiating the relationship

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^q_{ik}g_{qj} + \Gamma^q_{kj}g_{qi}\,,$$

with respect to x^l and using the fact that $g_{rs} = \delta^s_r$, we derive the relation

$$\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} = \frac{\partial \Gamma^j_{ik}}{\partial x^l} + \frac{\partial \Gamma^i_{kj}}{\partial x^l} + \Gamma^q_{ik}\Gamma^j_{ql} + \Gamma^q_{kj}\Gamma^i_{lq} + \Delta_{ijkl}\,,$$

where

$$\Delta_{ijkl} = \sum_{q=1}^{n}\left(\Gamma^q_{ki}\Gamma^q_{lj} + \Gamma^q_{kj}\Gamma^q_{li}\right),$$

so that $\Delta_{ijkl} = \Delta_{jikl}$.

We then apply the equality of mixed partial derivatives and rearrange the resulting equation, canceling Δ_{ijkl}.

PROBLEM 6.9. Prove that the parity of the permutation $(i_1,\ldots,i_k,j_1,\ldots,j_{n-k})$ differs from that of $(j_1,\ldots,j_{n-k},i_1,\ldots,i_k)$ by the factor $(-1)^{k(n-k)}$.

Solution: Starting with i_k, we can move each of the i_j to the right, just past j_{n-k} by $n-k$ interchanges of adjacent pairs. That is, each of the following k transformations changes the parity of the permutation by a factor of $(-1)^{n-k}$:

$$\begin{aligned}
(i_1,\ldots,i_k,j_1,\ldots,j_{n-k}) &\to (i_1,\ldots,i_{k-1},j_1,\ldots,j_{n-k},i_k)\\
(i_1,\ldots,i_{k-1},j_1,\ldots,j_{n-k},i_k) &\to (i_1,\ldots,i_{k-2},j_1,\ldots,j_{n-k},i_{k-1},i_k)\\
\vdots\ &\to\ \vdots\\
(i_1,j_1,\ldots,j_{n-k},i_2,\ldots,i_k) &\to (j_1,\ldots,j_{n-k},i_1,\ldots,i_k)\,.
\end{aligned}$$

Hence the first and the last differ in parity by a factor of $(-1)^{k(n-k)}$.

PROBLEM 6.10. Show that in normal coordinates on the 2-sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, the metric coefficients are

$$
\begin{aligned}
g_{11}(x,y) &= \frac{y^2\Bigl(\varphi\bigl(\frac{x^2+y^2}{r_0^2}\bigr)\Bigr)^2 + x^2\Bigl(\psi\bigl(\frac{x^2+y^2}{r_0^2}\bigr)\Bigr)^2}{x^2+y^2} + \frac{x^2}{r_0^2}\Bigl(\varphi\Bigl(\frac{x^2+y^2}{r_0^2}\Bigr)\Bigr)^2,\\
g_{22}(x,y) &= \frac{x^2\Bigl(\varphi\bigl(\frac{x^2+y^2}{r_0^2}\bigr)\Bigr)^2 + y^2\Bigl(\psi\bigl(\frac{x^2+y^2}{r_0^2}\bigr)\Bigr)}{x^2+y^2} + \frac{y^2}{r_0^2}\Bigl(\varphi\Bigl(\frac{x^2+y^2}{r_0^2}\Bigr)\Bigr)^2,\\
g_{12}(x,y) &= \frac{\Bigl(\Bigl(\psi\bigl(\frac{x^2+y^2}{r_0^2}\bigr)\Bigr)^2 - \Bigl(\varphi\bigl(\frac{x^2+y^2}{r_0^2}\bigr)\Bigr)^2\Bigr)xy}{x^2+y^2} + \frac{xy}{r_0^2}\Bigl(\varphi\Bigl(\frac{x^2+y^2}{r_0^2}\Bigr)\Bigr)^2.
\end{aligned}
$$

Here the functions $\varphi(t) = \sin(\sqrt{t})/\sqrt{t}$ and $\psi(t) = \cos(\sqrt{t})$ are needed only for non-negative values of t, namely $t = (x^2+y^2)/r_0^2$. They satisfy the easily established relations $t\bigl(\varphi(t)\bigr)^2 + \bigl(\psi(t)\bigr)^2 = 1$, $\varphi'(t) = \frac{1}{2}\bigl(\psi(t) - \varphi(t)\bigr)$, and $\psi'(t) = -\frac{1}{2}\varphi(t)$. It is easy to see that $g_{11}(x,y) \to 1$, $g_{22}(x,y) \to 1$, and $g_{12}(x,y) \to 0$ as $(x,y) \to (0,0)$. Thus, suppressing the argument $(x^2+y^2)/r_0^2$ of φ and ψ, since $\varphi \to 1$ and $\psi \to 1$ as $(x,y) \to (0,0)$, we find

$$
\begin{aligned}
0 \le |g_{11}(x,y) - 1| &= \Bigl|\frac{y^2(\varphi^2-1) + x^2(\psi^2-1)}{x^2+y^2} + \frac{x^2}{r_0^2}\varphi^2\Bigr|\\
&\le \max\bigl(|\varphi^2-1|, |\psi^2-1|\bigr) + \frac{x^2}{r_0^2}\varphi^2 \to 0\,,\\
0 \le |g_{22}(x,y) - 1| &= \Bigl|\frac{x^2(\varphi^2-1) + y^2(\psi^2-1)}{x^2+y^2} + \frac{y^2}{r_0^2}\varphi^2\Bigr|\\
&\le \max\bigl(|\varphi^2-1|, |\psi^2-1|\bigr) + \frac{y^2}{r_0^2}\varphi^2 \to 0\,,\\
0 \le |g_{12}(x,y)| &= \Bigl|\frac{(\psi^2-\varphi^2)xy}{x^2+y^2} + \frac{xy}{r_0^2}\varphi^2\Bigr| \le \frac{1}{2}\bigl|\psi^2-\varphi^2\bigr| + \frac{|xy|}{r_0^2}\varphi^2 \to 0\,.
\end{aligned}
$$

Convert these expressions to those given in the text (Example 6.2).

Deduce as a consequence that the element of surface area in these coordinates is

$$\sqrt{g_{11}g_{22} - g_{12}^2} = \frac{r_0}{r}\sin\Bigl(\frac{r}{r_0}\Bigr) \approx 1 - \frac{1}{6}\frac{r^2}{r_0^2}\,.$$

Use this density function (not the approximation) to show that the area of the spherical cap centered at $(0,0,r_0)$ and whose boundary lies at geodesic distance s from this point has area $4\pi r_0^2 \sin^2(s/2r_0)$. In particular, the upper hemisphere, for which $s = \pi r_0/2$ has area $2\pi r_0^2$, as it ought to.

Solution: Most of this is routine computation. The metric coefficients are easily reduced to those in the text by the substitution $x = r\cos\theta$, $y = r\sin\theta$, and the definitions of $\varphi(t)$ and $\psi(t)$. Likewise, the expression for the element of surface area is merely a matter of inserting the definition of g_{ij} and using trigonometric identities. Finally, the area of the spherical cap is just a matter of computing the integral

$$r_0\int_0^{2pi}\int_0^s \sin\Bigl(\frac{r}{r_0}\Bigr)\,dr\,d\theta = 2\pi r_0^2\Bigl(1 - \cos\Bigl(\frac{s}{r_0}\Bigr)\Bigr)\,.$$

PROBLEM 6.11. Prove the following simple lemma:

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be manifolds of the same dimension n with coordinate systems $(x^1, \ldots, x^n)$ at point $P_1 \in \mathfrak{M}_1$ corresponding to parameter value $(0, \ldots, 0)$ and $P_2 \in \mathfrak{M}_2$ also corresponding to parameter value $(0, \ldots, 0)$. Let the metric coefficients of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be $g_{ij}(x^1, \ldots, x^n)$ and $h_{ij}(x^1, \ldots, x^n)$ respectively, $i, j = 1, \ldots, n$. If the Maclaurin series of g_{ij} and h_{ij} have the same coefficients up to order 2—that is, if $h_{ij}(x^1, \ldots, x^n) = g_{ij}(x^1, \ldots, x^n) + O\bigl(((x^1)^2 + \cdots + (x^n)^2)^{3/2}\bigr)$—then the curvature of $\mathfrak{M}_1$ at P_1 equals the curvature of $\mathfrak{M}_2$ at P_2.

As a corollary, when one is computing the curvature of a manifold at a given point from the metric coefficients, these coefficients can be replaced by their Taylor expansions through quadratic terms.

Solution: The curvature is determined entirely by the values of the metric coefficients and their first and second derivatives at the base value of the Maclaurin expansions.

PROBLEM 6.12. Prove that the Laplace–Beltrami operator in cylindrical and spherical coordinates on $\mathbb{R}^3$ is just the form normally given for the Laplacian.

Solution: We give the details just for spherical coordinates; cylindrical coordinates are easier, especially since polar coordinates in the plane were discussed in the text itself.

The parametrization is

$$\boldsymbol{r}(\rho, \theta, \varphi) = \rho \cos\theta \, \sin\varphi \, \boldsymbol{i} + \rho \sin\theta \, \sin\varphi \, \boldsymbol{j} + \rho \cos\varphi \, \boldsymbol{k} \,,$$

The three basic tangent vectors are

$$\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial \rho} &= \cos\theta \, \sin\varphi \, \boldsymbol{i} + \sin\theta \, \sin\varphi \, \boldsymbol{j} + \cos\varphi \, \boldsymbol{k} \,, \\
\frac{\partial \boldsymbol{r}}{\partial \theta} &= -\rho \sin\theta \, \sin\varphi \, \boldsymbol{i} + \rho \cos\theta \, \sin\varphi \, \boldsymbol{j} \\
\frac{\partial \boldsymbol{r}}{\partial \varphi} &= \rho \cos\theta \, \cos\varphi \, \boldsymbol{i} + \rho \sin\theta \, \cos\varphi \, \boldsymbol{j} - \rho \sin\varphi \, \boldsymbol{k}, .
\end{aligned}$$

Here the longitude angle θ can be restricted to the range $-\pi < \theta < \pi$, and the co-latitude angle φ to $0 < \varphi < \pi$. Neither of these angles is defined at $\rho = 0$, and so we also need to restrict ρ to the range $0 < \rho < \infty$. What we have left is the unit ball with the half-plane $y \leq 0$ removed. On that domain, we can calculate the Laplace–Beltrami operator, as follows:

The matrix of metric coefficients is

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 \sin^2\varphi & 0 \\ 0 & 0 & \rho^2 \end{pmatrix} .$$

Thus an orthonormal basis for the tangent space is

$$\boldsymbol{u}_1 = \frac{\partial}{\partial \rho} \,, \quad \boldsymbol{u}_2 = \frac{1}{\rho \sin\varphi} \frac{\partial}{\partial \theta} \,, \quad \boldsymbol{u}_3 = \frac{1}{\rho} \frac{\partial}{\partial \varphi} \,.$$

The dual basis of the cotangent space is

$$\boldsymbol{v}_1 = d\rho \,, \quad \boldsymbol{v}_2 = \rho \sin\varphi \, d\theta \,, \quad \boldsymbol{v}_3 = \rho \, d\varphi \,.$$

From these correspondences, we easily see the following relations:

$$
\begin{aligned}
\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 &= \rho \sin\varphi \, d\rho \wedge d\theta \,, \\
\boldsymbol{v}_3 \wedge \boldsymbol{v}_1 &= \rho \, d\varphi \wedge d\rho \,, \\
\boldsymbol{v}_2 \wedge \boldsymbol{v}_3 &= \rho^2 \sin\varphi \, d\theta \wedge d\varphi \,, \\
\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \boldsymbol{v}_3 &= \rho^2 \sin\varphi \, d\rho \wedge d\theta \wedge d\varphi \,, \\
d\rho \wedge d\theta &= \frac{1}{\rho\sin\varphi} \, \boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \,, \\
d\varphi \wedge d\rho &= \frac{1}{\rho} \, \boldsymbol{v}_3 \wedge \boldsymbol{v}_1 \,, \\
d\theta \wedge d\varphi &= \frac{1}{\rho^2\sin\varphi} \, \boldsymbol{v}_2 \wedge \boldsymbol{v}_3 \,, \\
*(P\,\boldsymbol{v}_1 + Q\,\boldsymbol{v}_2 + R\,\boldsymbol{v}_3) &= P\,\boldsymbol{v}_2 \wedge \boldsymbol{v}_3 + Q\,\boldsymbol{v}_3 \wedge \boldsymbol{v}_1 + R\,\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \\
&= \rho^2 \sin\varphi \, P \, d\theta \wedge d\varphi + \rho \, Q \, d\varphi \wedge d\rho + \rho \sin\varphi \, R \, d\rho \wedge d\theta \,, \\
*(A\,\boldsymbol{v}_2 \wedge \boldsymbol{v}_3 + B\,\boldsymbol{v}_3 \wedge \boldsymbol{v}_1 + C\,\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 &= A\,\boldsymbol{v}_1 + B\,\boldsymbol{v}_2 + C\,\boldsymbol{v}_3 \\
&= A\,d\rho + \rho\sin\varphi \, B \, d\theta + \rho \, C \, d\varphi \,, \\
*F\,\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \boldsymbol{v}_3 &= F \,, \\
*(P\,d\rho + Q\,d\theta + R\,d\varphi) &= *\Big(P\,\boldsymbol{v}_1 + \frac{1}{\rho\sin\varphi}\,Q\,\boldsymbol{v}_2 + \frac{1}{\rho}\,R\,\boldsymbol{v}_3\Big) \\
&= P\,\boldsymbol{v}_2 \wedge \boldsymbol{v}_3 + \frac{1}{\rho\sin\varphi}\,Q\,\boldsymbol{v}_3 \wedge \boldsymbol{v}_1 + \frac{1}{\rho}\,R\,\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \\
&= \rho^2 \sin\varphi \, P \, d\theta \wedge d\varphi + \frac{1}{\sin\theta}\,Q\,d\varphi \wedge d\rho + \sin\varphi \, R \, d\rho \wedge d\theta \\
*(A\,d\theta \wedge d\varphi + B\,d\varphi \wedge d\rho + C\,d\rho \wedge d\theta) &= *\Big(\frac{1}{\rho^2\sin\varphi}\,A\,\boldsymbol{v}_2 \wedge \boldsymbol{v}_3 + \frac{1}{\rho}\,B\,\boldsymbol{v}_3 \wedge \boldsymbol{v}_1 + \frac{1}{\rho\sin\varphi}\,C\,\boldsymbol{v}_1 \wedge \boldsymbol{v}_2\Big) \\
&= \frac{1}{\rho^2\sin\varphi}\,A\,\boldsymbol{v}_1 + \frac{1}{\rho}\,B\,\boldsymbol{v}_2 + \frac{1}{\rho\sin\varphi}\,C\,\boldsymbol{v}_3 \\
&= \frac{1}{\rho^2\sin\varphi}\,A\,d\rho + \sin\varphi\,B\,d\theta + \frac{1}{\sin\varphi}\,C\,d\varphi \,, \\
*F\,d\rho \wedge d\theta \wedge d\varphi &= *\Big(\frac{1}{\rho^2\sin\varphi}F\,\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \boldsymbol{v}_3\Big) = \frac{1}{\rho^2\sin\varphi}F \,.
\end{aligned}
$$

Since $\delta f = 0$ for 0-forms f, we have

$$
\begin{aligned}
\nabla^2 f &= \delta(df) = *\big(\big(d\big(*(df)\big)\big) \\
&= *\bigg(d\Big(*\Big(\frac{\partial f}{\partial \rho}\,d\rho + \frac{\partial f}{\partial \theta}\,d\theta + \frac{\partial f}{\partial \varphi}\,d\varphi\Big)\Big)\bigg) \\
&= *\bigg(d\Big(\rho^2\sin\varphi\frac{\partial f}{\partial \rho}\,d\theta \wedge d\varphi + \frac{1}{\sin\varphi}\,\frac{\partial f}{\partial \theta}\,d\varphi \wedge d\rho + \sin\varphi\,\frac{\partial f}{\partial \varphi}\,d\rho \wedge d\theta \\
&= *\bigg(\frac{\partial}{\partial \rho}\,\Big(\rho^2\sin\varphi\frac{\partial f}{\partial \rho}\Big) + \frac{1}{\sin\varphi}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial}{\partial \varphi}\Big(\sin\varphi\frac{\partial f}{\partial \varphi}\Big)\bigg)\,d\rho \wedge d\theta \wedge d\varphi \\
&= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\Big(\rho^2\frac{\partial f}{\partial \rho}\Big) + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial f}{\partial \theta}^2 + \frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial \varphi}\Big(\sin\varphi\frac{\partial f}{\partial \varphi}\Big) \,.
\end{aligned}
$$

PROBLEM 6.13. Show that if $f(x^1, \ldots, x^n)$ is a C^2-function of compact support on $\mathbb{R}^n$, then

$$\int\limits_{\mathbb{R}^n} f(\boldsymbol{x})\nabla^2 f(\boldsymbol{x})\, d\boldsymbol{x} = -\sum_{k=1}^{n} \int\limits_{\mathbb{R}^n} \Big(\frac{\partial f}{\partial x^k}\Big)^2 \, d\boldsymbol{x}\,,$$

and that this expression is negative unless $f(\boldsymbol{x}) \equiv 0$.

Solution: This is just a matter of integrating by parts in each term of the sum

$$\int\limits_{\mathbb{R}^n} f(\boldsymbol{x})\nabla^2 f(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{k=1}^{n} \int\limits_{\mathbb{R}^n} f(\boldsymbol{x})\frac{\partial^2 f}{\partial (x^k)^2}\, d\boldsymbol{x}\,.$$

Without loss of generality, consider just the term where $i = n$. We then have

$$\int\limits_{\mathbb{R}^n} f(\boldsymbol{x})\frac{\partial^2 f}{\partial (x^n)^2}\, d\boldsymbol{x} = \int\limits_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} f(\boldsymbol{x})\frac{\partial^2 f}{\partial (x^n)^2}\, dx^n\, d\tilde{\boldsymbol{x}}\,,$$

where $d\tilde{\boldsymbol{x}} = dx^1 \cdots dx^{n-1}$. If we now integrate by parts, taking $u = f(\boldsymbol{x})$, $dv = \partial^2 f/\partial (x^n)^2$, we get $du = (\partial f/\partial x^n)\, dx^n$ and $v = \partial f/\partial x^n$. Since $f(\boldsymbol{x})$ has compact support, $u = 0$ when $x^n = \pm\infty$. The result now follows.

PROBLEM 6.14. Consider a surface in $\mathbb{R}^3$ consisting of the points $\big(x, y, z(x, y)\big)$. Let $x^1 = x$ and $x^2 = y$. First show that the Christoffel symbols for this surface have the simple expression

$$\Gamma^k_{ij} = \frac{\frac{\partial z}{\partial x^k}\,\frac{\partial^2 z}{\partial x^i\,\partial x^j}}{1 + \sum\limits_{l=1}^{2}\big(\frac{\partial z}{\partial x^l}\big)^2}\,.$$

Then show that the Hessian is given by

$$H_f(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i,j=1}^{2}\left(\frac{\Big(1 + \sum\limits_{l=1}^{2}\big(\frac{\partial z}{\partial x^l}\big)^2\Big)\frac{\partial^2 f}{\partial x^i\,\partial x_j} - \sum\limits_{l=1}^{2}\frac{\partial z}{\partial x^l}\,\frac{\partial f}{\partial x^l}\,\frac{\partial^2 z}{\partial x^i\,\partial x^j}}{1 + \sum\limits_{l=1}^{2}\big(\frac{\partial z}{\partial x^l}\big)^2}\right) u^i v^j\,.$$

Deduce as a corollary that

$$H_z(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i,j=1}^{2}\frac{\frac{\partial^2 z}{\partial x^i\,\partial x^j}\, u^i v^j}{1 + \sum\limits_{l=1}^{2}\big(\frac{\partial z}{\partial x^l}\big)^2}\,.$$

Thus the Hessian of a function z regarded as a function on the surface that is its graph is just the ordinary Hessian of z as a function on $\mathbb{R}^2$ divided by the square of the area density of the surface. In particular, at a point where the surface is tangent to the xy-plane, it coincides with the ordinary Hessian of z.

Solution: Although the computation is easy, *Mathematica* will instantly produce the Christoffel coefficients. All you have to do is get the metric coefficients, which

are

$$\begin{aligned} g_{11} &= 1+\Big(\frac{\partial z}{\partial x^1}\Big)^2\,,\\ g_{12}=g_{21} &= \frac{\partial z}{\partial x^1}\,\frac{\partial z}{\partial x^2}\,,\\ g_{22} &= 1+\Big(\frac{\partial z}{\partial x^2}\Big)^2\,, \end{aligned}$$

so that

$$\begin{aligned} g^{11} &= \frac{1+\big(\frac{\partial z}{\partial x^2}\big)^2}{1+\big(\frac{\partial z}{\partial x^1}\big)^2+\big(\frac{\partial z}{\partial x^2}\big)^2}\,,\\ g^{12}=g^{21} &= \frac{-\frac{\partial z}{\partial x^1}\,\frac{\partial z}{\partial x^2}}{1+\big(\frac{\partial z}{\partial x^1}\big)^2+\big(\frac{\partial z}{\partial x^2}\big)^2}\,,\\ g^{22} &= \frac{1+\big(\frac{\partial z}{\partial x^1}\big)^2}{1+\big(\frac{\partial z}{\partial x^1}\big)^2+\big(\frac{\partial z}{\partial x^2}\big)^2}\,, \end{aligned}$$

Suppressing the denominator $1+\big(\frac{\partial z}{\partial x^1}\big)^2+\big(\frac{\partial z}{\partial x^2}\big)^2$, we do indeed find that the numerator in the fraction that is Γ^k_{ij} is

$$\frac{\partial z}{\partial x^k}\,\frac{\partial^2 z}{\partial x^i\,\partial x^j}\,,$$

as asserted.

Since the Hessian is given by

$$H_f(\boldsymbol{u},\boldsymbol{v})=\Big(\frac{\partial^2 f}{\partial x^i\,\partial x^j}-\Gamma^k_{ij}\frac{\partial f}{\partial x^k}\Big)u^i v^j\,.$$

it follows easily that

$$H_z(\boldsymbol{u},\boldsymbol{v})=\frac{\frac{\partial^2 z}{\partial x^i\,\partial x^j}u^i v^j}{1+\Big(\frac{\partial z}{\partial x^1}\Big)^2+\Big(\frac{\partial z}{\partial x^2}\Big)^2}$$

Problem 6.15. We proved Corollary 6.3 by applying Eq. (81) to the second term in the expression

$$R^r_{prq}=\frac{\partial\Gamma^r_{pq}}{\partial u^r}-\frac{\partial\Gamma^r_{rp}}{\partial u^q}\,.$$

Apply that equation instead to the first term and show that

$$R^r_{prq}=-\frac{3}{2}g^{rm}\frac{\partial g_{rm}}{\partial u^p\,\partial u^q}\,.$$

Solution: The application of Eq. (81) yields the equation

$$R^r_{prq}=-\frac{\partial\Gamma^r_{qr}}{\partial u^p}-2\frac{\partial\Gamma^r_{rp}}{\partial u^q}\,.$$

But, as noted in the proof of Corollary 6.3, in normal coordinates at the origin,

$$\frac{\partial\Gamma^r_{qr}}{\partial u^p}=\frac{1}{2}g^{mr}\frac{\partial g_{mr}}{\partial u^p\,\partial u^q}\,.$$

Since this expression is symmetric in p and q, we can apply it also to the other term, getting the desired result.

PROBLEM 6.16. When the 2-sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ is parameterized by longitude and latitude coordinates, that is, by the mapping $(\theta, \varphi) \mapsto (\cos\theta\, \cos\varphi, \sin\theta\, \cos\varphi, \sin\varphi)$, the orthogonally invariant measure on it is $d\sigma_2(\theta, \varphi) = \cos\varphi\, d\theta\, d\varphi$. Prove that

$$\int\limits_{\mathbb{S}^2} (\xi^1)^2\, d\sigma_2 = \int\limits_{\mathbb{S}^2} (\xi^2)^2\, d\sigma_2 = \int\limits_{\mathbb{S}^2} (\xi^3)^2\, d\sigma_2 = \frac{4\pi}{3}\,.$$

(This is the special case of Theorem 6.13 that occurs when $f(\xi) = (\xi \cdot \boldsymbol{u})^2$ for the unit vectors $\boldsymbol{u} = (1, 0, 0)$, $\boldsymbol{u} = (0, 1, 0)$, and $\boldsymbol{u} = (0, 0, 1)$, since $\omega_2 = 4\pi$.)

Solution: The problem is to show that

$$\frac{4\pi}{3} = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\varphi \cos^2\theta\, d\varphi\, d\theta =$$
$$= \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\varphi \sin^2\theta\, d\varphi\, d\theta = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\varphi \cos\varphi\, d\varphi\, d\theta\,.$$

Each of these results is an elementary computation in calculus.

PROBLEM 6.17. Let $\varepsilon > 0$. Define two functions $f_\varepsilon(x)$ and $g_\varepsilon(x)$ on $[1, \infty)$ as follows:

$$f\varepsilon(x) = \begin{cases} \frac{n\varepsilon}{2} \sin^2\left(\pi n^3(x - n)\right), & \text{if } n \le x \le n + \frac{1}{n^3}\,, \quad n = 1, 2, 3, \ldots, \\ 0\,, & \text{otherwise} \end{cases}\,,$$

$$g_\varepsilon(x) = \frac{\pi^2\varepsilon}{24} + \int_1^x f_\varepsilon(s)\, ds\,.$$

Show that $f_\varepsilon(x) \ge 0$, and that $f_\varepsilon(n + 1/(2n^3)) = n\varepsilon/2$, so that $f_\varepsilon(n + 1/(2n^3)) \to \infty$ as $n \to \infty$. Thus, in particular, $f_\varepsilon(x)$ is not bounded. Then show that $g_\varepsilon(x)$ is an increasing function of x, and that for all x,

$$0 < \frac{\pi^2\varepsilon}{24} \le g_\varepsilon(x) \le \frac{\pi^2\varepsilon}{12} < \varepsilon$$

By letting ε tend to zero, show that $g_\varepsilon(x)$ can be made arbitrarily small while its derivative $g'_\varepsilon(x) = f_\varepsilon(x)$ remains unbounded.

Solution: The fact that $f_\varepsilon \ge 0$ is trivial, since it is a positive constant times a squared function. That fact also implies immediately that g_ε is an increasing function. The value of f_ε at $x = n + 1/(2n^3)$ is also an immediate computation. The first inequality to be proved is just the fact that $g(x)$ is an increasing (actually, non-decreasing) function of x, and the last one is a matter of adding up the terms of the series

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n\varepsilon}{2} \int_n^{n+\frac{1}{n^3}} \sin^2(\pi n^3(x - n))\, dx &= \sum_{n=1}^{\infty} \frac{n\varepsilon}{2} \int_0^{\frac{1}{n^3}} \sin^2(\pi n^3 x)\, dx \\ &= \sum_{n=1}^{\infty} \frac{n\varepsilon}{2\pi n^3} \int_0^{\pi} \sin^2 y\, dy = \frac{1}{4} \sum_{n=1}^{\infty} \frac{\varepsilon}{n^2} = \frac{\pi^2\varepsilon}{24}\,, \end{aligned}$$

so that $g_\varepsilon(x) \le \frac{\pi^2\varepsilon}{24} + \frac{\pi^2\varepsilon}{24} = \frac{\pi^2\varepsilon}{12}$, as asserted. Hence $g_\varepsilon(x)$ tends uniformly to 0 as $\varepsilon \to 0$, even though the derivative $g'_\varepsilon = f_\varepsilon$ remains unbounded.

PROBLEM 6.18. Verify that the covariant derivative $\nabla_{\boldsymbol{v}}$ has the derivation property $\nabla_{\boldsymbol{v}}(f\boldsymbol{u}) = \nabla_{\boldsymbol{v}} f \boldsymbol{u} + f \nabla_{\boldsymbol{v}} \boldsymbol{u}$.

Solution: This again is a routine computation. Just let $\boldsymbol{v} = v^i\partial/\partial x^i$ and $\boldsymbol{u} = u^i\partial/\partial x^i$. Then

$$\begin{aligned}
\nabla_{\boldsymbol{v}}(f\boldsymbol{u}) &= v^j\Big(\frac{\partial(fu^l)}{\partial x^j} + fu^k\Gamma^l_{jk}\Big)\frac{\partial}{\partial x^l}\\
&= \bigg(fv^j\Big(\frac{\partial u^l}{\partial x^j} + u^k\Gamma^l_{jk}\Big) + v^ju^l\frac{\partial f}{\partial x^j}\bigg)\frac{\partial}{\partial x^l}\\
&= f\nabla_{\boldsymbol{v}}(\boldsymbol{u}) + \Big(v^j\frac{\partial f}{\partial x^j}\Big)\Big(u^l\frac{\partial}{\partial x^l}\Big)\\
&= f\nabla_{\boldsymbol{v}}(\boldsymbol{u}) + \nabla_{\boldsymbol{v}}(f)\boldsymbol{u}\,.
\end{aligned}$$

PROBLEM 6.19. Show that the computed sectional curvature $\varkappa(\boldsymbol{u},\boldsymbol{v})$ does not change if the vector $\boldsymbol{u}$ is replaced by $a\boldsymbol{u} + b\boldsymbol{v}$, for any nonzero a and any b. Thus, the sectional curvature depends only on the plane spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution: The argument here is identical to the proof of Theorem 6.4.

PROBLEM 6.20. Show that the length of the geodesic $\boldsymbol{\gamma}_{\boldsymbol{u}}(t)$ used in defining the exponential mapping is $|\boldsymbol{u}|$.

Solution: As was shown in the discussion leading up to the definition of the exponential mapping the length of the path from $\boldsymbol{\gamma}_{\boldsymbol{u}}(0)$ to $\boldsymbol{\gamma}_{\boldsymbol{u}}(t)$ is just the arc length $s = t|\boldsymbol{u}|$ corresponding to the value of t. In particular, when $t = 1$, that length is $|\boldsymbol{u}|$.

PROBLEM 6.21. The three-sphere $\mathbb{S}^3$ of radius 1, whose sectional curvature we have computed, is an excellent example of a Lie group. It consists of the points in $\mathbb{R}^4$ that we may identify with points in space-time, calling them $T = (t^0; t^1, t^2, t^3)$. Then

$$\mathbb{S}^3 = \{T : (t^0)^2 + (t^1)^2 + (t^2)^2 + (t^3)^2 = 1\}\,.$$

What makes this manifold a Lie group is the group operation of *quaternion multiplication*. If we identify the quaternion T with the formal sum of a real number and a vector in $\mathbb{R}^3$, say $T = t^0 + \boldsymbol{\tau}$, where t^0 is identified with the quaternion $(t^0, 0, 0, 0)$ and $\boldsymbol{\tau} = t^1\boldsymbol{i} = t^2\boldsymbol{j} + t^3\boldsymbol{k}$ is identified with the quaternion $(0, t^1, t^2, t^3)$, the group operation is quaternion multiplication: If $S = s^0 + \boldsymbol{\sigma}$, then $ST = (s^0t^0 - \boldsymbol{\sigma}\cdot\boldsymbol{\tau}) + (s^0\boldsymbol{\tau} + t^0\boldsymbol{\sigma} + \boldsymbol{\sigma}\times\boldsymbol{\tau})$.

Verify that $\mathbb{S}^3$ is a group under the operation of quaternion multiplication with identity $I = 1 + \mathbf{0}$.

Solution: It is a routine computation, based on the well-known identity $(\boldsymbol{u}\cdot\boldsymbol{v})^2 + |\boldsymbol{u}\times\boldsymbol{v}|^2 = |\boldsymbol{u}|^2|\boldsymbol{v}|^2$, to show that $|ST| = |S|\,|T|$, so that $\mathbb{S}^3$ is closed under the operation of multiplication. It is even simpler to verify that I is an identity for this operation. Its associativity is a more tedious computation, but can be verified using *Mathematica*.

Like a complex number, each quaternion $T = t^0 + \boldsymbol{\tau}$ has a "quaternion conjugate" $\bar{T} = t^0 - \boldsymbol{\tau}$, and the quaternion product $T\bar{T}$ is the quaternion $|T|^2 + \mathbf{0}$, identified with the real number that is its square-norm: $|T|^2 = T\bar{T} = (t^0)^2 + (t^1)^2 + (t^2)^2 + (t^3)^2 + \mathbf{0}$, which, when $T \in \mathbb{S}^3$, is always equal to $I = 1 + \mathbf{0}$. It follows easily that $\bar{T}$ is the inverse of T when $T \in \mathbb{S}^3$.

PROBLEM 6.22. The exponential mapping at the point $(1; 0, 0, 0)$ in the group of unit quaternions is given by the analog of the mapping used in Example 6.2 on the sphere $\mathbb{S}^2$, namely

$$\exp(\boldsymbol{x}) = \psi\big(|\boldsymbol{x}|^2\big) + \varphi\big(|\boldsymbol{x}|^2\big)\boldsymbol{x}$$

when rectangular coordinates $\boldsymbol{x} = (x, y, z)$ are used on the tangent space $\mathbb{R}^3$. Here $\psi(t) = \cos\big(\sqrt{t}\big)$ and $\varphi(t) = \sin\big(\sqrt{t}\big)/\sqrt{t}$, as in Example 6.2. This expression shows that there is no singularity at the origin. In spherical coordinates (ρ, φ, θ), where there is a breakdown of the parametrization at that point, we have $\boldsymbol{x} = \Big(\rho\cos\varphi\cos\theta, \rho\cos\varphi\sin\theta, \rho\sin\varphi\Big)$, and

$$\exp(\boldsymbol{x}) = \cos(\rho) + \sin(\rho)\big(\cos\varphi\cos\theta\,\boldsymbol{i} + \cos\varphi\sin\theta\,\boldsymbol{j} + \sin\varphi\,\boldsymbol{k}\big)\,.$$

Verify that for a unit vector $\boldsymbol{x}$ the mapping $s \mapsto \exp(s\boldsymbol{x})$ is a geodesic with s as arc length, and that

$$\exp(s\boldsymbol{x})\exp(t\boldsymbol{x}) = \exp\big((s+t)\boldsymbol{x}\big)\,.$$

The product on the left is the quaternion product. This equation justifies the name *exponential mapping.*

Solution: The assertion of the exponential property is a straightforward application of the definition of quaternion multiplication. In rectangular coordinates

$$\begin{aligned}\exp(s\boldsymbol{x})\exp(t\boldsymbol{x}) &= \Big(\cos\big(s|\boldsymbol{x}|\big) + \frac{\sin\big(s|\boldsymbol{x}|\big)}{|\boldsymbol{x}|}\boldsymbol{x}\Big)\Big(\cos\big(t|\boldsymbol{x}|\big) + \frac{\sin\big(t|\boldsymbol{x}|\big)}{|\boldsymbol{x}|}\boldsymbol{x}\Big)\\ &= \cos\big(s|\boldsymbol{x}|\big)\cos\big(t|\boldsymbol{x}|\big) - \sin\big(s|\boldsymbol{x}|\big)\sin\big(t|\boldsymbol{x}|\big) +\\ &+ \Big(\cos\big(t|\boldsymbol{x}|\big)\frac{\sin\big(s\boldsymbol{x}|\big)}{|\boldsymbol{x}|} + \sin\big(s|\boldsymbol{x}|\big)\frac{\cos\big(s|\boldsymbol{x}|\big)}{|\boldsymbol{x}|}\Big)\boldsymbol{x} =\\ &= \cos\big((s+t)|\boldsymbol{x}|\big) + \frac{\sin\big((s+t)|\boldsymbol{x}|\big)}{|\boldsymbol{x}|}\boldsymbol{x} = \exp\big((s+t)\boldsymbol{x}\big)\,.\end{aligned}$$

Likewise, the assertion that s is arc length in the mapping $\exp(s\boldsymbol{x})$ when $\boldsymbol{x}$ is a unit vector is a straightforward computation, since

$$\begin{aligned}\Big|\frac{d}{ds}\big(\exp(s\boldsymbol{x})\big)\Big|^2 &= \big|-|\boldsymbol{x}|\sin\big(s|\boldsymbol{x}|\big) + \cos\big(s|\boldsymbol{x}|\big)\boldsymbol{x}\big|^2\\ &= |\boldsymbol{x}|^2\big(\sin^2\big(s|\boldsymbol{x}|\big) + \cos^2\big(s|\boldsymbol{x}|\big)\big) = |\boldsymbol{x}|^2\,.\end{aligned}$$

As for the verification that the stated mapping $\exp(s\boldsymbol{x})$ is a geodesic with arc length s, this is best done by considering the mapping into the parameter space: $s \mapsto \boldsymbol{x}(s) = (s, \varphi, \theta)$, where φ and θ are fixed. Then $(x^i)'(s) = \delta^i_1$ and $(x^i)''(s) \equiv 0$. Then the only non-zero term in the geodesic equation

$$(x^i)'' + \Gamma^i_{jk}(x^j)'(x^k)' = 0$$

occurs when $j = k = 1$, and the equation simply says

$$\Gamma^i_{11} = 0\,,$$

for $i = 1, 2, 3$. That equation is easily verified in spherical coordinates. We can adapt *Mathematica* Notebook 10 for the purpose. All we have to do is compute as far as the Christoffel symbols, and *Mathematica* reveals that only nine of these are non-zero, namely $\Gamma^1_{22} = -\cos\rho\sin\rho$, $\Gamma^1_{33} = -\cos\rho\sin\rho\cos^2\varphi$, $\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \cot\rho$, $\Gamma^3_{23} = \Gamma^3_{32} = -\tan\varphi$, and $\Gamma^2_{33} = \cos\varphi\sin\varphi$.

PROBLEM 6.23. Let $\boldsymbol{\gamma}(t)$ be a smooth curve in a manifold, $0 \le t \le t_0$. For each tangent vector $\boldsymbol{w}$ at $\boldsymbol{\gamma}(0)$, let $\boldsymbol{w}(t)$ be the parallel transport of $\boldsymbol{w}$ along $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}(t)$. Show that the mapping $\boldsymbol{w} \mapsto \boldsymbol{w}(t)$ is a linear transformation and that it preserves the length of a vector. That is the expression $g_{ij}\bigl(\boldsymbol{\gamma}(t)\bigr)w^i(t)w^j(t)$ is constant. (It follows that this mapping preserves the standard inner product on the tangent space, so that any two vectors map to two other vectors having the same lengths and forming the same angle.)

Solution: Fix t, and let $T(\boldsymbol{w}) = \boldsymbol{w}(t)$. The function $\boldsymbol{w}(t)$ is defined as the unique solution of the initial-value problem given by the system of equations

$$\begin{aligned} (w^i)'(t) &= A^i(t)\bigl(\boldsymbol{\gamma}'(t), \boldsymbol{w}(t)\bigr)\,, \\ w^i(0) &= w^i\,, \end{aligned}$$

$i = 1, 2, \ldots, n$, where $A^i(t)$ is the bilinear functional on the tangent space at $\boldsymbol{\gamma}(t)$ whose matrix in the standard basis $\{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$ is

$$\begin{pmatrix} -\Gamma^i_{11} & \cdots & -\Gamma^i_{1n} \\ \vdots & \ddots & \vdots \\ -\Gamma^i_{n1} & \cdots & -\Gamma^i_{nn} \end{pmatrix}$$

The fact that T is a linear operator follows from the bilinearity of the functional $A^i(t)$: For any two vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ in the tangent space at $\boldsymbol{\gamma}(0)$ and any two scalars a and b, the tangent vector $\boldsymbol{w}(t) = a\boldsymbol{w}_1(t) + b\boldsymbol{w}_2(t) = aT(\boldsymbol{w}_1) + bT(\boldsymbol{w}_2)$ satisfies

$$\begin{aligned} (w^i)'(t) &= a(w^i_1)'(t) + b(w^i_2)'(t) \\ &= aA^i(t)\bigl(\boldsymbol{\gamma}'(t), \boldsymbol{w}_1(t)\bigr) + bA^i(t)\bigl(\boldsymbol{\gamma}'(t), \boldsymbol{w}_2(t)\bigr) \\ &= A^i(t)\bigl(\boldsymbol{\gamma}'(t), a\boldsymbol{w}_1(t) + b\boldsymbol{w}_2(t)\bigr) \\ &= A^i(t)\bigl(\boldsymbol{\gamma}'(t), \boldsymbol{w}(t)\bigr)\,, \\ w^i(0) &= aw^i_1(0) + bw^i_2(0) \\ &= aw^i_1 + bw^i_2\,. \end{aligned}$$

By definition of parallel transport, this set of equations means that $\boldsymbol{w}(t) = T(a\boldsymbol{w}_1 + b\boldsymbol{w}_2)$, and thus $T(a\boldsymbol{w}_1 + b\boldsymbol{w}_2) = aT(\boldsymbol{w}_1) + bT(\boldsymbol{w}_2)$, which is the definition of linearity.

The linearity now being established, let $L(t)$ denote the length of $\boldsymbol{w}(t)$, that is, $L(t) = g_{ij}\bigl(\boldsymbol{\gamma}(t)\bigr)w^i(t)w^j(t)$. All we have to show is that $L'(t) = 0$, and this is a routine computation. We have

$$\begin{aligned} L'(t) &= \frac{\partial g_{ij}}{\partial x^k}(\gamma^k)'(t)w^i(t)w^j(t) + 2g_{ij}\bigl(\boldsymbol{\gamma}(t)\bigr)w^i(t)(w^j)'(t) \\ &= \frac{\partial g_{ij}}{\partial x^k}(\gamma^k)'(t)w^i(t)w^j(t) - 2(\gamma^k)'(t)g_{ij}\bigl(\boldsymbol{\gamma}(t)\bigr)w^i(t)w^l(t)\Gamma^j_{kl}\bigl(\boldsymbol{\gamma}(t)\bigr) \\ &= (\gamma^k)'(t)w^i(t)\Bigl(\frac{\partial g_{ij}}{\partial x^k}w^j(t) - 2g_{ij}\bigl(\boldsymbol{\gamma}(t)\bigr)w^l(t)\Gamma^j_{kl}\bigl(\boldsymbol{\gamma}(t)\bigr)\Bigr)\,. \end{aligned}$$

Everything now comes down to the definition of the Christoffel symbol:

$$\Gamma^j_{kl} = \frac{1}{2}g^{jn}\Bigl(\frac{\partial g_{kn}}{\partial x^l} + \frac{\partial g_{ln}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^n}\Bigr)$$

The sum $g_{ij}\Gamma^j_{kl}$ becomes

$$\frac{1}{2}\delta^n_i\Big(\frac{\partial g_{kn}}{\partial x^l}+\frac{\partial g_{ln}}{\partial x^k}-\frac{\partial g_{kl}}{\partial x^n}\Big)=\frac{1}{2}\Big(\frac{\partial g_{ki}}{\partial x^l}+\frac{\partial g_{li}}{\partial x^k}-\frac{\partial g_{kl}}{\partial x^i}\Big)$$

We then find

$$L'(t)=(\gamma^k)'(t)\bigg(w^i(t)w^j(t)\frac{\partial g_{ij}}{\partial x^k}-w^i(t)w^l(t)\Big(\frac{\partial g_{ki}}{\partial x^l}+\frac{\partial g_{li}}{\partial x^k}-\frac{\partial g_{kl}}{\partial x^i}\Big)\bigg)\,.$$

If we replace the dummy index of summation l in the second term here by j, we get

$$L'(t)=(\gamma^k)'(t)\bigg(w^i(t)w^j(t)\frac{\partial g_{ij}}{\partial x^k}-w^i(t)w^j(t)\Big(\frac{\partial g_{ki}}{\partial x^j}+\frac{\partial g_{ji}}{\partial x^k}-\frac{\partial g_{kj}}{\partial x^i}\Big)\bigg)\,.$$

Since the terms inside the large brackets now obviously cancel in pairs, we see that $L'(t)\equiv 0$, as required. (The equality

$$w^iw^j\Big(\frac{\partial g_{ki}}{\partial x^j}-\frac{\partial g_{ji}}{\partial x^i}\Big)=0$$

holds because i and j are dummy indices of summation, so that if they are interchanged, the expression is unchanged. But obviously, interchanging i and j causes this expression to reverse its sign. Hence it must equal 0.)

CHAPTER 7

The Geometrization of Gravity

PROBLEM 7.1. Imagine a tunnel dug all the way through the earth around one of its diameters. (Idealize the earth as a perfect sphere and imagine its atmosphere has disappeared, so that the inside of the tunnel is a perfect vacuum.) What would happen to a particle dropped down the very center of the tunnel? (Use Newtonian reasoning.)

Solution: Let us simplify this by assuming a constant density ρ for the earth and neglecting the miniscule portion of it that is missing, having been excavated for the tunnel. Since a spherical shell exerts no gravitational attraction at points inside it, the force per unit mass when the particle is at distance r from the center is

$$r''(t) = -\frac{4\pi}{3}\,\frac{G\rho\big(r(t)\big)^3}{\big(r(t)\big)^2} = -\frac{4\pi G\rho}{3}r(t)\,.$$

The equation satisfied is thus that of the harmonic oscillator:

$$r''(t) + \omega^2 r(t) = 0\,,$$

where the frequency ω is

$$\omega = \sqrt{\frac{4\pi G\rho}{3}}\,.$$

(You can verify that ω has the physical dimension of a frequency.)

Thus, the more dense the sphere, the faster the oscillation. As for the amplitude R of the oscillation, if we suppose that $r(0) = r_e$, where r_e is the radius of the earth, and $r'(0) = 0$ (the particle is dropped from rest) we find that $R = r_e$, and the equation of motion is $r(t) = r_e \cos\omega t$.

PROBLEM 7.2. Show that a plane whose first fundamental form is given by Eq. (92) has curvature $\kappa(r)$ depending only on r and given by

$$\kappa(r) = -\frac{c^2 r_s}{r^3}\,.$$

(Thus, somewhat surprisingly, given that the parametrization has a singularity at $r = r_s$, there is no singularity in the curvature at that point.)

Solution: Once again, there is no need to sweat and strain, since *Mathematica* can do this problem instantly. Here is the code, in which r_s is denoted `rs`:

```
x = {t, r}; gsub = {{1 - rs/r, 0}, {0, -r/(c^2 (r - rs))}};
u = {1, 0}; v = {0, 1}; gsup = Inverse[gsub];
Γ = FullSimplify[Table[(1/2) Sum[gsup[[i, l]] (D[gsub[[j, l]], x[[k]]]
+
```

```
D[gsub[[l, k]], x[[j]]] - D[gsub[[j, k]], x[[l]]]), {l, 1, 2}], {i, 1,
2}, {j, 1, 2}, {k, 1, 2}]];
Riem = FullSimplify[Table[Sum[gsub[[m, i]] (D[Γ[[m, l, j]], x[[k]]] -
D[Γ[[m, k, j]], x[[l]]] +
Sum[Γ[[n, l, j]] Γ[[m, k, n]] - Γ[[n, k, j]] Γ[[m, l, n]], {n, 1, 2}]),
{m, 1, 2}],
{l, 1, 2}, {k, 1, 2}, {j, 1, 2}, {i, 1, 2}]];
κ = FullSimplify[Sum[Riem[[i, j, k, l]] u[[i]] v[[j]] u[[k]] v[[l]],
{l, 1, 2}, {k,1, 2}, {j, 1, 2}, {i, 1, 2}] /Det[gsub]]
```

Problem 7.3. Show that the Laplace–Beltrami operator on the plane with first fundamental form (92) is

$$\nabla^2 f = \frac{r}{r-r_s}\frac{\partial^2 f}{\partial t^2} - \frac{c^2(r-r_s)}{r}\frac{\partial^2 f}{\partial r^2} - \frac{c^2 r_s}{r^2}\frac{\partial f}{\partial r}\,.$$

Solution: We can continue to use the *Mathematica* code from the previous problem, just remembering that the Laplace–Beltrami operator is given by

$$\nabla^2 f = g^{il}\frac{\partial^2 f}{\partial x^i\,\partial x^l} + \Big(\frac{g^{kl}}{2\det(M)}\frac{\partial\big(\det(M)\big)}{\partial x^l} + \frac{\partial g^{kl}}{\partial x^l}\Big)\frac{\partial f}{\partial x^k}\,.$$

The code for the Laplace–Beltrami operator is

```
LB[f_] := Sum[gsup[[i,l]] D[D[f,x[[i]]], x[[l]]], {i, 1, 2}, {l, 1, 2}]
+
Sum[((gsup[[k,l]]/(2 Det[gsub])) D[Det[gsub], x[[l]] ] + D[gsup[[k, l]],
x[[l]]]) D[f, x[[k]]], {k, 1, 2}, {l, 1, 2}]
```

and it does indeed yield exactly the stated result, when you ask for `LB[f[t,r]]`.

Problem 7.4. Assuming $\ell = 1$, confirm Hilbert's statement that a body falling toward a black hole has acceleration toward the black hole if its speed $v = r'(t)$ satisfies $|v| < c(r-r_s)/(\sqrt{3}r)$ and away from it if $|v| > c(r-r_s)/(\sqrt{3}r)$.

Solution: We saw that the acceleration changes sign where $r = 3r_s$. Given the values in the text for v and a, this means $|v| = 2c/(3\sqrt{3})$, and this is precisely the value of $c(r-r_s)/(\sqrt{3}r)$ when $r = 3r_s$.

Problem 7.5. Show that the space-time given by the Gödel metric has no singularities. Also show that the Gaussian curvature of the section tangent to any plane containing the z-axis is 0, that of the sections tangent to the tx and ty planes is ω^2, and that of the section tangent to the xy-plane is $3\omega^2$. Show finally that the scalar curvature is $2\omega^2$.

Solution: The first assertion is merely a matter of inspecting the coefficients in the metric

$$ds^2 = \frac{1}{2\omega^2}\Big(d\tau^2 + 2e^{\xi}\,d\tau\,d\eta - d\xi^2 + \frac{e^{2\xi}}{2}\,d\eta^2 - d\zeta^2\Big)$$

and observing that they are all entire functions of their arguments. For the rest, we once again appeal to a general *Mathematica* notebook. First we define the parameters τ, ξ, η, and ζ, the matrix of metric coefficients `gsub` and its inverse `gsup`, the tableau Γ of Christoffel symbols, the natural inner product on the tangent

space `dot`, tangent vectors `t,x,y,z` along the t-, x-, y-, and z-axes, a general tangent vector `v`, the Riemann tensor `Riem`, and finally its operator form `R[u,v,w]`.

```
x = {τ, ξ, η, ζ};
gsub = (1/(2 ω^2)) {{1,0,Exp[ξ], 0}, {0,-1,0,0},
{Exp[ξ],0,Exp[2 ξ]/2,0}, {0,0,0,-1}}; gsup = Inverse[gsub];
Γ = FullSimplify[Table[(1/2) Sum[gsup[[i,q]] (D[gsub[[j,q]],x[[k]]]
+D[gsub[[q,k]],x[[j]]] - D[gsub[[j,k]],x[[q]]]),{q,1,4}],
{i,1,4}, {j,1,4}, {k,1,4}]];
dot[y_,z_]:= Sum[gsub[[i,j]] y[[i]] z[[j]], {i,1,4}, {j,1,4}];
ut = {1,0,0,0}; ux = {0,1,0,0}; uy = {0,0,1,0}; uz = {0,0,0,1};
v = {a,b,c,d};
Riem = FullSimplify[Table[D[Γ[[i,j,l]],x[[k]]] - D[Γ[[i,j,k]],x[[l]]]
+ Sum[Γ[[n,j,l]] Γ[[i,k,n]] - Γ[[n,j,k]] Γ[[i,l,n]],
{n,1,4}], {i,1,4}, {j,1,4}, {k,1,4}, {l,1,4}]];
R[u_, v_, w_]:= Table[Sum[Riem[[i,j,k,l]] u[[k]] v[[l]] w[[j]],
{j,1,4}, {k,1,4}, {l,1,4}], {i,1,4}]
```

Having these tools out our disposal, we set about computing the stated sectional curvatures. With self-explanatory notation, we set

```
ktx = FullSimplify[dot[ut, R[ut,ux,ux]]/(dot[ut,ut] dot[ux,ux]
- dot[ut,ux]^2)]
kty = FullSimplify[dot[ut,R[ut,uy,uy]]/(dot[ut,ut] dot[uy,uy]
- dot[ut,uy]^2)]
kz = FullSimplify[dot[v,R[v,uz,uz]]/(dot[v,v] dot[uz,uz]
- dot[v,uz]^2)]
kxy = FullSimplify[dot[ux,R[ux,uy,uy]]/(dot[ux,ux] dot[uy,uy]
- dot[ux,uy]^2)]
```

The outputs of these last four commands are respectively ω^2, ω^2, 0, and $3\omega^2$, as asserted.

Finally, we set up the Ricci tensor and find the scalar curvature. (In order to preserve the traditional notation for the latter, we use the letter `R` for it, even though that conflicts with our notation for the operator form of the Riemann curvature tensor. If we asked *Mathematica* afterwards to go back and compute the sectional curvatures, it would give 0 as the result, due to this redefinition. It does no harm here, however, since we are finished computing the sectional curvatures.)

```
Ric = FullSimplify[Table[Sum[Riem[[i,j,i,l]],
{i,1,4}], {j,1,4}, {l,1,4}]];
R = FullSimplify[Sum[gsup[[i,j]] Ric[[i,j]], {i,1,4}, {j,1,4}]]
```

The output of this last command is $2\omega^2$, as was claimed, and we are now finished.

Problem 7.6. Prove that the manifold $\mathfrak{M}$ given by Gödel's original metric is a *homogeneous space*, in the sense that, for any two points P and Q in $\mathfrak{M}$, there is an isometry $T_{PQ} : \mathfrak{M} \to \mathfrak{M}$ (a one-to-one mapping of $\mathfrak{M}$ onto itself that preserves the metric) such that $T_{PQ}(P) = Q$. To do this, show that the mapping

$$\begin{aligned} T_{PQ}(t; x, y, z) &= (t + q^1 - p^1; x + q^2 - p^2, e^{p^2-q^2}(y - p^3) + q^3, z + q^4 - p^4) \\ &= (\tau; \xi, \eta, \zeta) \end{aligned}$$

is a one-to-one diffeomorphic (infinitely differentiable, in fact, analytic) isometry of $\mathfrak{M}$ onto itself mapping $P = (p^1; p^2, p^3, p^4)$ to $Q = (q^1; q^2, q^3, q^4)$.

Solution: There is no difficulty showing that P maps to Q. We need only show that the metric is preserved.

In this transformation, we have $d\tau = dt$, $d\xi = dx$, $d\eta = e^{p^2-q^2}dy$, and $d\zeta = dz$. We thus have to show that

$$ds^2 = 2e^{\xi}\,d\tau\,d\eta - \frac{e^{2\xi}}{2}\,d\eta^2 = 2e^x\,dt\,dy - \frac{e^{2x}}{2}\,dy^2\,.$$

This is a routine computation.

PROBLEM 7.7. Those who appreciate the power and beauty of the theory of analytic functions of a complex variable may yearn to see this theory extended to three-dimensional space $\mathbb{R}^3$. It is possible to create such an extension? A number of considerations come to mind, algebraic, geometric, and analytic.

The algebraic consideration is that $\mathbb{R}^3$ is *not* a field, but the plane $\mathbb{R}^2$ *is*. (The latter is the field of complex numbers.) Indeed, it was his attempt to find a suitable definition of multiplication for elements of $\mathbb{R}^3$ that led Hamilton to discover quaternions, which are a multiplication operation on $\mathbb{R}^4$. Thus, there is an algebraic barrier to such an extension.

The geometric barrier is even more formidable. Analytic function theory provides a plethora of conformal mappings. By the Riemann mapping theorem, any simply-connected subset of the plane that has at least two boundary points can be conformally mapped onto the unit disk. In contrast, only a very restricted class of mappings of open sets of $\mathbb{R}^3$ can be conformal.

Nevertheless, analysis can still forge ahead and define a mapping $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$, given by $\boldsymbol{F}(x, y, z) = u(x, y, z)\boldsymbol{i} + v(x, y, z)\boldsymbol{j} + w(x, y, z)\boldsymbol{k}$, to be *conjugate-analytic* if $\nabla \times \boldsymbol{F} = \boldsymbol{0}$ and $\nabla \cdot \boldsymbol{F} = 0$. In particular, the Newtonian gravitational force is a conjugate-analytic function on $\mathbb{R}^3 \setminus \{\boldsymbol{0}\}$. Show that, when $w \equiv 0$ and u and v are independent of z (so that $\boldsymbol{F}$ maps the plane into itself), these equations reduce to the Cauchy–Riemann equations for the function $f(z) = f(x+iy) = u(x, y) - iv(x, y)$, namely $\partial u/\partial x = -\partial v/\partial y$ and $\partial u/\partial y = \partial v/\partial x$. Thus if $\boldsymbol{F} = u\boldsymbol{i} + v\boldsymbol{j}$ is identified with the complex function $f(x + iy) = u(x, y) + iv(x, y)$, it is the conjugate of an analytic function. Use these equations to show that u and v are harmonic functions, that is, $\nabla^2 u = 0 = \nabla^2 v$.

Also show that the components $u(x, y, z)$, $v(x, y, z)$, and $w(x, y, z)$ of a conjugate-analytic function $\boldsymbol{F} = u(x, y, z)\,\boldsymbol{i} + v(x, y, z)\,\boldsymbol{j} + w(x, y, z)\,\boldsymbol{k}$ are harmonic functions.

Solution: The first statement is a routine computation. If $\boldsymbol{F} = u(x, y)\,\boldsymbol{i} + v(x, y)\,\boldsymbol{j}$, the equation $\nabla \times \boldsymbol{F} = \boldsymbol{0}$ says that

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0\,.$$

The equation $\nabla \cdot \boldsymbol{F} = 0$ says

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,.$$

These are precisely the Cauchy–Riemann equations for the function $f(x + iy) = u(x, y) - iv(x, y)$.

In general, conjugate-analyticity is equivalent to the four equations

$$\begin{aligned}
\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} &= 0\,, \\
\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} &= 0\,, \\
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0\,, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0\,.
\end{aligned}$$

If we differentiate the first equation with respect to x, the second with respect to y and the third with respect to z, we find that

$$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial^2 v}{\partial x\,\partial z} = \frac{\partial^2 w}{\partial x\,\partial y}\,.$$

Now the second, third, and fourth equations imply that

$$\begin{aligned}
\frac{\partial^2 u}{\partial z^2} &= \frac{\partial^2 w}{\partial x\,\partial z}\,, \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 v}{\partial x\,\partial y}\,, \\
\frac{\partial^2 u}{\partial x^2} &= -\frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial^2 w}{\partial x\,\partial z}\,.
\end{aligned}$$

Adding these equations, we get $\nabla^2 u = 0$. By symmetry, we also have $\nabla^2 v = 0 = \nabla^2 w$.

APPENDIX 1

Hyperbolic Trigonometry

Problem 1.1. The formula for the angle of parallelism provides a way of determining the curvature of the particular hyperbolic plane one happens to be on. Show that the distance C for which the angle of parallelism is half of a right angle is

$$C = k \ln\left(\sqrt{2} + 1\right).$$

Thus, if (1) physical space is hyperbolic and (2) it actually is possible to determine the angle of parallelism experimentally—it probably isn't—this equation can be solved for the radius of curvature $k\sqrt{-1}$. What velocity does this natural unit of length correspond to when translated into the space of relativistic velocities?

Solution: The hyperbolic tangent function and its inverse are related as follows:

$$y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\,; \quad x = \ln\sqrt{\frac{1+y}{1-y}}\,.$$

Our formula for the angle of parallelism ξ at distance x is

$$\cot\xi = \sinh(x/k)\,.$$

Taking $\xi = \pi/4$ and $x = C$, we find that $C = k\ln(1+\sqrt{2})$.

By direct computation, the corresponding velocity u is

$$\begin{aligned} u &= c\tanh(C/k) \\ &= \frac{(1+\sqrt{2}) - 1/(1+\sqrt{2})}{(1+\sqrt{2}) + 1/(1+\sqrt{2})}c \\ &= \frac{1+\sqrt{2}}{2+\sqrt{2}}c \\ &= \frac{\sqrt{2}}{2}c = \frac{c}{\sqrt{2}}\,. \end{aligned}$$

Problem 1.2. In a manuscript that was not published during his lifetime, Gauss defined the area of a hyperbolic triangle to be its defect times k^2, where k is the radius of curvature of hyperbolic space. He then gave a formula for the least upper bound of the areas of all finite triangles in terms of C^2, where C is the absolute unit of length in Problem 1.1. What is that least upper bound?

Solution: Given that the least upper bound of the defects of all triangles is π, it is evident that the upper bound for the areas is πk^2, which is equivalent to

$$\pi k^2 = \frac{\pi C^2}{\left(\ln(1+\sqrt{2})\right)^2}\,.$$

This is the formula given by Gauss.

Although the problem does not ask this, the reader may be curious to know if the area formula can be obtained from the metric on the pseudo-sphere that we have introduced. That is indeed possible, and the details now follow.

A right triangle T_0 having one acute angle equal to $\pi/4$ with the leg adjacent to that angle just slightly smaller than the length C, is a very long, but finite triangle whose third angle is very small. If we reflect T_0 about the leg whose length is slightly smaller than C, the triangle T_0 and its image form an isosceles right triangle T_1 with very long legs (each equal to the hypotenuse of T_0 and (of course) an even longer hypotenuse. The area of T_1 is twice that of T_0. If we then reflect T_1 about one of its legs, we get an equilateral triangle T_2 whose area is four times the area of T_0 and whose three equal angles are very small. The limiting area of T_0 as the original leg increases to C will be four times the limit of the area of T_0, and will be the least upper bound for the areas of all finite triangles. Hence it would suffice to find the area of the figure that is the limit of T_0 as the given leg approaches C.

This, as it happens, is not difficult to do. If we look at the geodesic whose polar equation is $r = kr_0/\sqrt{k^2 - r_0^2\theta^2}$ on the pseudo-sphere parameterized by

$$(r, \theta) \mapsto \bigl(r, \theta, z(r, \theta)\bigr)\,,$$

where

$$z(r, \theta) = k\Biggl(\ln\Bigl(\frac{k}{r} - \sqrt{\bigl(\frac{k}{r}\bigr)^2 - 1}\Bigr) - \sqrt{1 - \bigl(\frac{r}{k}\bigr)}\Biggr)\,,$$

then the metric is given by

$$ds^2 = \frac{k^2}{r^2}\,dr^2 + r^2\,d\theta^2\,,$$

and the element of area is

$$dA = \sqrt{EG - R^2}\,dr\,d\theta = k\,dr\,d\theta\,.$$

If we take $r_0 = k/\sqrt{2}$, we find that the portion of the geodesic between the parameter values $(r_0, 0) = (k/\sqrt{2}, 0)$ and $\bigl(k, \sqrt{(k/r_0)^2 - 1}\bigr) = (k, 1)$ has length exactly C, and so we can easily compute the area of this triangle as

$$k\int_0^1\int_0^{k/\sqrt{2-\theta^2}} dr\,d\theta = k^2\int_0^1 \frac{1}{\sqrt{2-\theta^2}}\,d\theta = \pi k^2/4\,.$$

The upper bound for the area of any finite triangle is thus four times this amount, which is πk^2, as defined by Gauss.

Problem 1.3. Prove that two lines in the hyperbolic plane can have at most one common perpendicular.

Solution: Suppose that two lines have two common perpendiculars. The portions of these lines between the common perpendiculars and the portions of the two common perpendiculars between the two lines then form a rectangle, that is, a quadrilateral having four right angles. This rectangle can be partitioned into two triangles by drawing a diagonal. Then the sum of the angles of at least one of those triangles must be at least two right angles, which is never the case in hyperbolic geometry.

Problem 1.4. Prove that a line and a boundary parallel to it cannot have a common perpendicular.

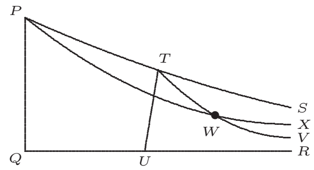

FIGURE 1. Impossibility of a common perpendicular to a pair of boundary parallel lines

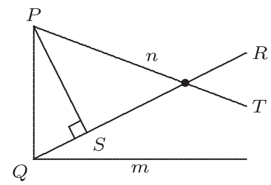

FIGURE 2. Boundary parallelism is a symmetric relationship.

Solution: Let PS and QR be boundary parallel rays, as shown in Fig. 1, and suppose that TU is a common perpendicular to them. Then TS is *not* the boundary parallel to UR. Draw the boundary parallel TV to QR from T on the side away from P, and let W be any point on this ray. Draw the line PX passing through P and W. Since it crosses TV at W, it remains above this boundary parallel to QR and hence does not intersect it. On the other hand, since the ray is inside the angle QPS, it *must* intersect QR. This contradiction shows that no such common perpendicular can exist. (A little analysis of the proof shows that the boundary parallel to QR from P is also the boundary parallel from T, since assuming otherwise leads to exactly the same figure In fact, we didn't actually use the full assumption that TU is perpendicular to PS in order to arrive at the contradiction, only the consequence of that assumption, that TS is not the boundary parallel to QR and also does not intersect it.)

PROBLEM 1.5. Prove that if (1) line n passes through a point P not on line m and is boundary parallel to m, and (2) Q is the foot of the perpendicular from P to m, then m is the boundary parallel to n on the same side of PQ on which n is boundary parallel to m.

Solution: This proposition asserts that boundary parallelism is a symmetric relationship. To prove it, let PT be the ray lying on the line n that is parallel to m, and let QR be any ray from Q inside the right angle at Q on the side of m containing P and on the side of PQ containing the boundary parallel ray (see Fig. 2). Since angle PQR is acute, the perpendicular from P to the line containing QR intersects QR in a point S, forming a right triangle with hypotenuse PQ. Thus PS is shorter than PQ, and so the angle of parallelism for the distance PS is larger than $\angle QPT$, which is the angle of parallelism for the distance PQ. But $\angle SPT < \angle QPT$, and therefore $\angle SPT$ is smaller than the angle of parallelism for the distance PS. It follows that PT intersects the ray QR.

PROBLEM 1.6. Derive the two kinds of geodesics on the pseudo-sphere from the element of arc length.

Solution: We recall that the element of arc length ds satisfies

$$ds^2 = \frac{k^2}{r^2}\,dr^2 + r^2\,d\theta^2\,.$$

Thus, if a path is parameterized by arc length, then

$$\begin{aligned} L &= \int_0^L \sqrt{\frac{k^2(r')^2}{r^2} + r^2\dot{\theta}^2}\,ds\,, \\ F(r, r', \theta') &= \sqrt{\frac{k^2(r')^2}{r^2} + r^2(\theta')^2} \equiv 1\,. \end{aligned}$$

If the path is a geodesic, by Euler's formula from the calculus of variations,

$$\frac{d}{ds}\Bigl(\frac{\partial F}{\partial \theta'}\Bigr) = \frac{\partial F}{\partial \theta} \equiv 0\,.$$

Since $F(r, r', \theta') \equiv 1$, this equation implies that for some constant K,

$$\theta' = \frac{K}{r^2}\,.$$

It follows similarly that

$$k^2(r')^2 = r^2 - K^2\,.$$

We remark first of all that this equation implies that K, which has the geometric dimension of length, cannot be larger than any value of r on the geodesic. When $r = K$, we have $r' = 0$, and therefore if the value K is one that r assumes on the geodesic, it is the minimum value. Since there must be such a minimum value unless r can approach zero along the geodesic (which can happen only if z approaches infinity), and since r' cannot vanish unless $r = K$, we shall take it as established that either $K = 0$ or $K = r_0$ is the minimum value of r along the geodesic.

To dispose of the first case immediately, we consider what happens if $K = 0$. In that case, θ is a constant, say θ_0. Then, depending on the orientation of the geodesic, we have

$$r' = \pm\frac{1}{k}r\,,$$

(the ambiguous sign is the same at all points of the geodesic, depending only on the direction in which length is regarded as positive) and so

$$r = r_0 e^{\pm s/k}\,,$$

where r_0 is the value of r at parameter value $s = 0$.

This means

$$s = \pm k \ln\Bigl(\frac{r}{r_0}\Bigr)\,,$$

and the length of a path between parameter values s_1 and s_2 is

$$L = |s_1 - s_2| = k\Bigl|\ln\Bigl(\frac{r_2}{r_0}\Bigr) - \ln\Bigl(\frac{r_1}{r_0}\Bigr)\Bigr| = k\Bigl|\ln\Bigl(\frac{r_2}{r_1}\Bigr)\Bigr|\,.$$

The element of arc length when this path is parameterized by r is

$$ds = \pm\frac{k}{r}\,dr\,.$$

We now consider the case $K = r_0 \neq 0$. Choosing an orientation for the geodesic, we can assume the differential equation

$$\frac{k\,dr}{\sqrt{r^2 - r_0^2}} = ds\,.$$

Either of the substitutions $r = r_0 \sec u$ or $r = r_0 \cosh u$ makes it possible to integrate the left-hand side, and so we find with either substitution (assuming u positive) that for some constant l,

$$s = l + k \ln\left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\right).$$

If we assume s measured from the point where r has the minimum value r_0, we have $l = 0$. In any case, the distance between two points with radial coordinates r_1 and r_2 is

$$s_2 - s_1 = k\left|\ln\left(\frac{r_2 + \sqrt{r_2^2 - r_0^2}}{r_1 + \sqrt{r_1^2 - r_0^2}}\right)\right|.$$

This geodesic starts and ends on the equatorial circle ($r = k$, which actually is not part of the open pseudo-hemisphere, but it is convenient to include it anyway). The minimum value r_0 and the value θ_0 at which it occurs can be determined from any two points on the geodesic. To do so we change variables from s to θ, obtaining the differential equation

$$\frac{dr}{d\theta} = \frac{r'}{\theta'} = \frac{r^2\sqrt{r^2 - r_0^2}}{kr_0}\,.$$

Thus we find that

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{kr_0\,dr}{r^2(\sqrt{r^2 - r_0^2})}\,.$$

The substitution $r = r_0 \sec\varphi$ turns this last integral into the simple integral

$$\int_0^{\varphi} \frac{k}{r_0}\cos(t)\,dt = \frac{k}{r_0}\sin\varphi = k\sqrt{\frac{1}{r_0^2} - \frac{1}{r^2}}\,.$$

As a result, we find

$$r = \frac{kr_0}{\sqrt{k^2 - r_0^2(\theta - \theta_0)^2}}\,.$$

This is the equation of the trace of the geodesic on the equatorial disk, from which the value of z corresponding to each value of the parameter θ is readily computed.

This geodesic intersects the equatorial circle $r = k$ at the two values $\theta = \theta_0 \pm \sqrt{(k^2/r_0^2) - 1}$. If r_0 is nearly equal to k, these values are very close together, and the geodesic does not go very high on the pseudo-hemisphere, but as $r_0 \downarrow 0$, θ may wind around the pseudo-hemisphere many times before reaching its maximum height (at θ_0) and then descending again to the equatorial circle.

Problem 1.7. For a hyperbolic right triangle having an acute angle θ with adjacent side X and hypotenuse H, show that

$$\cos\theta = \frac{\tanh(X/k)}{\tanh(H/k)}\,.$$

Solution: This is very straightforward algebra, carried out by using the identities $\cosh(H/k) = \cosh(X/k)\cosh(Y/k)$ (where Y is the side opposite $\angle\theta$), $\sin\theta = \sinh(Y/k)/\sinh(H/k)$, $\cos^2\theta + \sin^2\theta = 1$, $\cosh^2(Z/k) - \sinh^2(Z/k) = 1$, and $\tanh^2(Z/k) + 1/\cosh^2(Z/k) = 1$.

Problem 1.8. We can develop analytic geometry for the hyperbolic plane of curvature $-k^2$ by considering a pair of mutually perpendicular axes labeled the X-axis and the Y-axis, as shown in Fig. 3. From a point in the upper half-plane, we drop lines perpendicular to the two axes and use the distances from the feet of those perpendiculars to the origin as the coordinates of the point. The resulting quadrilateral has three right angles (and hence necessarily an acute angle at the point being labeled) and is usually called a *Lambert quadrilateral* after Johann Heinrich Lambert (1728–1777), although such quadrilaterals had been studied centuries earlier by Thabit ibn-Qurra (826–901). Show that the conversions from polar coordinates (R, θ) to rectangular coordinates (X, Y) and vice versa are given by

$$\begin{aligned}
X &= k \operatorname{arctanh}\Big(\tanh(R/k)\cos\theta\Big)\,,\\
Y &= k \operatorname{arctanh}\Big(\tanh(R/k)\sin\theta\Big)\,,\\
R &= k \operatorname{arctanh}\sqrt{\tanh^2(X/k) + \tanh^2(Y/k)}\,,\\
\theta &= \arctan\Big(\frac{\tanh(Y/k)}{\tanh(X/k)}\Big)\,.
\end{aligned}$$

Solution: All of these results are consequence of the trigonometric relations in a right triangle. If the side adjacent to acute angle θ is a, the side opposite, is b, and the hypotenuse is c, we have the fundamental relations

$$\begin{aligned}
\cosh(c/k) &= \cosh(a/k)\cosh(b/k)\,,\\
\sin\theta &= \frac{\sinh(b/k)}{\sinh(c/k)}\,,\\
\cos\theta &= \frac{\tanh(a/k)}{\tanh(c/k)}\,,\\
\tan\theta &= \frac{\tanh(b/k)}{\sinh(a/k}\,.
\end{aligned}$$

The first of these is a consequence of the law of cosines applied to the special case when the angle is a right angle, and it was established in Chapter 1. The second is the law of sines, also proved in Chapter 1 and also applied here for the special case when one of the two angles is a right angle. The other two follow from the first two and the fundamental identities $\sin^2\theta + \cos^2\theta = 1$ and $\tan(\theta) = \sin(\theta)/\cos(\theta)$.

To take the cosine first, we have

$$\begin{aligned}
\cos^2\theta &= 1-\sin^2\theta \\
&= 1-\frac{\sinh^2(b/k)}{\sinh^2(c/k)} \\
&= \frac{\sinh^2(c/k)-\sinh^2(b/k)}{\sinh^2(c/k)} \\
&= \frac{\cosh^2(c/k)-\cosh^2(b/k)}{\sinh^2(c/k)} \\
&= \frac{\cosh^2(c/k)-\cosh^2(c/k)/\cosh^2(a/k)}{\sinh^2(c/k)} \\
&= \frac{\big(\cosh^2(a/k)-1\big)/\cosh^2(a/k)}{\tanh^2(c/k)} \\
&= \frac{\tanh^2(a/k)}{\tanh^2(c/k)}\,.
\end{aligned}$$

Since θ is an acute angle and a, c, and k are all positive numbers, we now have only to take the square root on both sides.

The fourth relation is now a consequence of the first three. From the second and third, we have

$$\begin{aligned}
\tan\theta &= \frac{\sin\theta}{\cos\theta} \\
&= \frac{\sinh(b/k)\tanh(c/k)}{\sinh(c/k)\tanh(a/k)} \\
&= \frac{\sinh(b/k)}{\cosh(c/k)\tanh(a/k)} \\
&= \frac{\sinh(b/k)}{\cosh(a/k)\cosh(b/k)\tanh(a/k)} \\
&= \frac{\tanh(b/k)}{\sinh(a/k)}\,.
\end{aligned}$$

Referring now to Fig. 3, we write

$$\begin{aligned}
\cos\theta &= \frac{\tanh(X/k)}{\tanh(R/k)}\,, \\
\sin\theta &= \cos\left(\frac{\pi}{2}-\theta\right) = \frac{\tanh(Y/k)}{\tanh(R/k)}\,, \\
\tan\theta &= \frac{\tanh(Y/k)}{\tanh(X/k)}\,.
\end{aligned}$$

All that remains now is to show that the expression for R is correct. But that also is easy, since

$$1=\cos^2\theta+\sin^2\theta=\cos^2\theta+\cos^2\left(\frac{\pi}{2}-\theta\right)^2=\frac{\tanh^2(X/k)+\tanh^2(Y/k)}{\tanh^2(R/k)}\,.$$

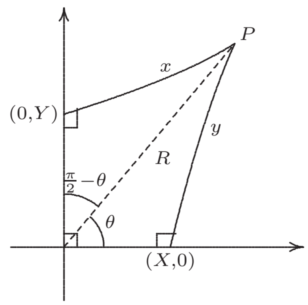

FIGURE 3. The basis of analytic geometry in the hyperbolic plane: A Thabit–Lambert quadrilateral. The point P has rectangular coordinates (X, Y) and polar coordinates (R, θ). The angle at P is acute, and sides x and y are longer than the sides opposite them.

PROBLEM 1.9. Show that the element of arc length in polar coordinates (R, θ) in the hyperbolic plane is

$$ds^2 = dR^2 + k^2 \sinh^2\left(\frac{R}{k}\right) d\theta^2 .$$

Solution: We imagine a triangle with vertex at the origin, and two sides of lengths R and $R + dR$ meeting there to form an angle $d\theta$. Then ds is the side opposite the angle $d\theta$. By the hyperbolic law of cosines and the combinatorial trick of writing $\cos(d\theta)$ as $1 - \big(1 - \cos(d\theta)\big)$, we find

$$\begin{aligned}\cosh\left(\frac{ds}{k}\right) &= \cosh\left(\frac{R}{k}\right)\cosh\left(\frac{R+dR}{k}\right) - \sinh\left(\frac{R}{k}\right)\sinh\left(\frac{R+dR}{k}\right)\cos(d\theta)\\ &= \cosh\left(\frac{R}{k}\right)\Bigg(\cosh\left(\frac{R}{k}\right)\cosh\left(\frac{dR}{k}\right) + \sinh\left(\frac{R}{k}\right)\sinh\left(\frac{dR}{k}\right)\Bigg)\\ &- \sinh\left(\frac{R}{k}\right)\Bigg(\sinh\left(\frac{R}{k}\right)\cosh\left(\frac{dR}{k}\right) + \cosh\left(\frac{R}{k}\right)\sinh\left(\frac{dR}{k}\right)\Bigg)\Big(1 - \big(1 - \cos(d\theta)\big)\Big).\end{aligned}$$

After we collect like terms and use the identity $\cosh^2(x) - \sinh^2(x) = 1$, this expression simplifies to

$$\begin{aligned}\cosh\left(\frac{ds}{k}\right) &= \cosh\left(\frac{dR}{k}\right) +\\ &+ \Bigg(\sinh^2\left(\frac{R}{k}\right)\cosh\left(\frac{dR}{k}\right) + \sinh\left(\frac{R}{k}\right)\cosh\left(\frac{R}{k}\right)\sinh\left(\frac{dR}{k}\right)\big(1 - \cos(d\theta)\big)\Bigg).\end{aligned}$$

If we now expand all the functions containing infinitesimals in Maclaurin series out to second order, recalling that

$$\begin{aligned}\cos x &= 1 - \frac{1}{2}x^2 + \cdots\\ \cosh x &= 1 + \frac{1}{2}x^2 + \cdots\\ \sinh x &= x + \frac{1}{6}x^3 + \cdots\end{aligned}$$

we have

$$1+\frac{1}{2}\Big(\frac{ds}{k}\Big)^2 = 1+\frac{1}{2}\Big(\frac{dR}{k}\Big)^2+\left(\sinh^2\left(\frac{R}{k}\right)\Big(1+\frac{1}{2}\Big(\frac{dR}{k}\Big)^2+\cosh\left(\frac{R}{k}\right)\Big(\frac{dR}{k}\Big)\Big)\right)\Big(\frac{1}{2}d\theta^2\Big)\,.$$

Canceling the term equal to 1 on both sides, then multiplying both sides by 2, and finally suppressing all infinitesimals of order higher than 2, we get

$$ds^2 = dR^2 + k^2\sinh^2\left(\frac{R}{k}\right)d\theta^2\,.$$

Problem 1.10. Deduce from the previous problem that this means that the circumference of a hyperbolic circle of radius R is $2\pi k\sinh\big(R/k\big)$, and that the element of area is

$$dA = k\sinh\left(\frac{R}{k}\right)dR\,d\theta\,,$$

so that the area of a circle of radius R is $2\pi k^2\Big(\cosh\big(R/k\big)-1\Big) = \pi\big(2k\sinh(R/(2k))\big)^2$.

Solution: The first part follows immediately upon taking $dR = 0$ and integrating ds in terms of θ from 0 to 2π. As for the second, we recall that when the element of length is given by the first fundamental form $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$, then the element of area is given by the form $dA = \sqrt{EG-F^2}$. In the present case, $E\equiv 1$, $F\equiv 0$, and $G = k^2\sinh^2(R/k)$, and so the computation is immediate.

Problem 1.11. Show that the geodesics in the disk of radius c with the relativistic velocity metric of Notebook 9, that is,

$$ds^2 = \frac{c^4}{(c^2-r^2)^2}\,dr^2 + \frac{c^2r^2}{c^2-r^2}\,d\theta^2,$$

are precisely the chords (not including their endpoints).

Solution: The geodesic problem (see Appendix 2) is to minimize the arc-length integral

$$\int ds = \int\sqrt{\frac{c^4}{(c^2-r^2)^2}(r')^2 + \frac{r^2c^2}{c^2-r^2}\,(\theta')^2}\,ds = \int F(r,\theta,r',\theta')\,ds\,,$$

where $F(s)\equiv 1$ when arc length s is used as parameter, as we always assume to begin with. We thus begin with the following pair of Euler equations:

$$\begin{aligned}\frac{d}{ds}\Big(\frac{\partial F}{\partial r'}\Big) &= \frac{\partial F}{\partial r}\,,\\ \frac{d}{ds}\Big(\frac{\partial F}{\partial \theta'}\Big) &= \frac{\partial F}{\partial \theta}\,.\end{aligned}$$

In the present case, after a small amount of algebraic manipulation, we get the equations

$$\begin{aligned} r'' + \frac{2r(r')^2}{c^2-r^2} - r(\theta')^2 &= 0\,,\\ \theta' &= k\,\frac{c^2-r^2}{c^2r^2}\,,\end{aligned}$$

where k is a constant. The second equation arises because the function $F(r,\theta,r',\theta')$ is independent of θ.

To find the geodesics, we need to express r as a function of θ, and that means replacing the independent variable s in the first of these equations with θ. To do so, we need to divide this equation by $(\theta')^2$. Therefore, we first need to dispose of the case when $k = 0$. Thus we now pose the initial-value problem for this second-order equation by assuming initial conditions $r(0) = r_0$, $r'(0) = r_1$, θ being given any constant value (since it does not occur in the equation anyway) and r_1 being non-zero except in the trivial case of a single-point "geodesic." The usual trick of setting $p = r'$, $r'' = dp/ds = p\,dp/dr$ yields the equation

$$\frac{dp}{p} + \frac{2r\,dr}{c^2 - r^2} = 0\,,$$

which is trivial to integrate and shows that $p = B(c^2 - r^2)$ for the constant $B = r_1/(c^2 - r_0^2)$. Then, replacing p by dr/ds, we get the differential equation

$$\frac{dr}{c^2 - r^2} = B\,ds\,,$$

which integrates easily to yield, after more algebraic manipulation,

$$r = c\Bigl(\frac{(c + r_0)e^{\beta s} - (c - r_0)}{(c + r_0)e^{\beta s} + (c - r_0)}\Bigr)\,,$$

where $\beta = 2cr_1/(c^2 - r_0^2)$. We see that $r \to \pm c$ as $s \to \pm\infty$, so that r traces a diameter of the disk as s varies over all real values. The case of constant θ is now complete, and we assume henceforth that $k \neq 0$.

In the general case, dividing the Euler equation on r by $(\theta')^2$ eventually yields the equation

$$\frac{d^2r}{d\theta^2} - \frac{2}{r}\Bigl(\frac{dr}{d\theta}\Bigr)^2 - r = 0\,.$$

Here the same trick used to calculate Newtonian orbits in Chapter 4, namely letting $r = 1/u$, converts this equation to the standard equation of an oscillator, that is,

$$\frac{d^2u}{d\theta^2} + u = 0\,,$$

whose general solution is

$$u = a\,\cos\theta + b\,\sin\theta\,,$$

Thus, in polar coordinates, the equation of a geodesic is

$$1 = ra\,\cos\theta + rb\,\sin\theta\,,$$

which in rectangular coordinates is just the equation of a straight line:

$$ax + by = 1\,.$$

PROBLEM 1.12. Verify that along the geodesic on the pseudo-hemisphere that projects into the equatorial plane as the curve whose polar equation is

$$r = \frac{kr_0}{\sqrt{k^2 - r_0^2(\theta - \theta_0)^2}}$$

the element of arc length is

$$ds = \frac{k^2 r_0}{k^2 - r_0(\theta - \theta_0)^2}\,d\theta\,,$$

where θ_0 is the value of θ at which the radial coordinate r has its minimum value (r_0) along the geodesic (which corresponds to the highest point on the geodesic itself).

Solution: It is clear that $r \geq r_0$, with equality only at $\theta = \theta_0$. The rest of the problem is merely tedious algebra, substituting the values of r and dr into the equation

$$ds^2 = \frac{k^2}{r^2}\, dr^2 + r^2\, d\theta^2 \,.$$

APPENDIX 2

Calculus of Variations. Geodesics

Problem 2.1. Prove the assertion in the text that if $f(t)$ is any continuous function on an interval $[a, b]$ such that

$$\int_a^b u(t)f(t)\,dt = 0$$

for all continuously differentiable functions $u(t)$ satisfying $u(a) = u(b) = 0$ or $u(a) = u'(a) = 0$, then $f(t) \equiv 0$ on that interval.

Solution: Suppose that $f(t) \not\equiv 0$ on the interval $[a, b]$. Since $f(t)$ is continuous, there is some interior point c in the interval for which $f(c) \neq 0$. (If $f(t) \equiv 0$ for $a < t < b$ and f is continuous, then $f(a) = 0 = f(b)$ also, and $f(t) \equiv 0$ on the closed interval $[a, b]$.) Replacing $f(t)$ by $-f(t)$ if necessary, we can assume that $f(c) > 0$. Then, since $f(t)$ is continuous, there is some interval (g, h) with $a < g < c < h < b$ such that $f(t) \geq f(c)/2 = \varepsilon$ for $g \leq t \leq h$. Define $u(t)$ on $[a, b]$ by

$$u(t) = \begin{cases} 0\,, & \text{if } a \leq t \leq g\,, \\ \sin^2\left(\frac{\pi(t-g)}{h-g}\right), & \text{if } g \leq t \leq h\,, \\ 0\,, & \text{if } h \leq t \leq b\,. \end{cases}$$

The function $u(t)f(t)$ is non-negative at all points, since $f(t) \geq \varepsilon$ at all points where $u(t)$ is non-zero, and $u(t)$ is non-negative at all points whatever. The derivative $u'(t)$ is continuous and equal to 0 on $[a, g] \cup [h, b]$, and so in fact $u(t)$ satisfies *both* conditions $u(a) = u'(a) = 0$ and $u(a) = u(b) = 0$. But

$$\int_a^b u(t)f(t)\,dt \geq \int_{a_1}^{b_1} u(t)f(t)\,dt\,,$$

where $a_1 = (5g + h)/6$ and $b_1 = (g + 5h)/6$. It is easy to see that $g < a_1 < b_1 < h$, and that $u(t) \geq 1/4$ on $[a_1, b_1]$. Thus,

$$\int_{a_1}^{b_1} u(t)f(t)\,dt \geq (b_1 - a_1)\varepsilon/4 = \frac{\varepsilon}{6}(h - g) > 0\,.$$

Thus, if $f(t)$ is *not* identically 0 on $[a, b]$, there is a function $u(t)$ that is continuously differentiable on $[a, b]$ and satisfies $u(a) = u'(a) = u(b) = 0$ and is such that

$$\int_a^b u(t)f(t)\,dt > 0\,.$$

Problem 2.2. For a particle moving at constant speed along a geodesic of the surface $F(x, y, z) = f(x, y) - z = 0$, show that the third component of its acceleration is given by

$$z'' = \frac{\Delta}{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,.$$

In other words, the acceleration is $-k\,\nabla F(x,y,z)$, where

$$k = \frac{\Delta}{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,.$$

Solution: This argument mostly repeats what was done in the text, where it was shown that

$$\begin{aligned} x'' &= \frac{-\frac{\partial f}{\partial x}\Delta}{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,, \\ y'' &= \frac{-\frac{\partial f}{\partial y}\Delta}{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,, \\ z'' &= \Delta + \frac{\partial f}{\partial x}\,x'' + \frac{\partial f}{\partial y}\,y''\,. \end{aligned}$$

where

$$\Delta = \frac{\partial^2 f}{\partial x^2}(x')^2 + 2\frac{\partial^2 f}{\partial x \partial y}x'\,y' + \frac{\partial^2 f}{\partial y^2}(y')^2\,.$$

By routine computation, then, substituting the values of x'' and y'' in the equation for z'', factoring out Δ, and canceling the squares of the two first-order partial derivatives, we get the required result.

PROBLEM 2.3. Show that the first fundamental form for the surface $F(x,y,z) = f(x,y) - z = 0$ is $E\,dx^2 + 2F\,dx\,dy + G\,dy^2$, where

$$\begin{aligned} E &= 1+\left(\frac{\partial f}{\partial x}\right)^2, \\ F &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\,, \\ G &= 1+\left(\frac{\partial f}{\partial y}\right)^2. \end{aligned}$$

Solution: This is again a routine computation, taking $\boldsymbol{r}(x,y) = \big(x, y, f(x,y)\big)$, so that

$$\begin{aligned} \frac{\partial \boldsymbol{r}}{\partial x} &= \left(1, 0, \frac{\partial f}{\partial x}\right), \\ \frac{\partial \boldsymbol{r}}{\partial y} &= \left(0, 1, \frac{\partial f}{\partial y}\right). \end{aligned}$$

The desired equations are now merely a matter of computing:

$$\begin{aligned} E &= \frac{\partial \boldsymbol{r}}{\partial x}\cdot\frac{\partial \boldsymbol{r}}{\partial x} = 1+\left(\frac{\partial f}{\partial x}\right)^2, \\ F &= \frac{\partial \boldsymbol{r}}{\partial x}\cdot\frac{\partial \boldsymbol{r}}{\partial y} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\,, \\ G &= \frac{\partial \boldsymbol{r}}{\partial y}\cdot\frac{\partial \boldsymbol{r}}{\partial y} = 1+\left(\frac{\partial f}{\partial y}\right)^2. \end{aligned}$$

PROBLEM 2.4. Show that the element of surface area dS on the surface $f(x,y)-z=0$ is

$$dS = \sqrt{1+\Big(\frac{\partial f}{\partial x}\Big)^2+\Big(\frac{\partial f}{\partial y}\Big)^2}\,dy\,dx.$$

Solution: We know that

$$dS = \Big|\frac{\partial \boldsymbol{r}}{\partial x}\times\frac{\partial \boldsymbol{r}}{\partial y}\Big|\,dy\,dx\,.$$

These partial derivatives were computed in the preceding problem. The required expression is now a simple matter of computing the cross product and taking its absolute value:

$$\begin{aligned}\frac{\partial \boldsymbol{r}}{\partial x}\times\frac{\partial \boldsymbol{r}}{\partial y} &= \Big(-\frac{\partial f}{\partial x},-\frac{\partial f}{\partial y},1\Big)\,,\\ \Big|\frac{\partial \boldsymbol{r}}{\partial x}\times\frac{\partial \boldsymbol{r}}{\partial y}\Big| &= \sqrt{1+\Big(\frac{\partial f}{\partial x}\Big)^2+\Big(\frac{\partial f}{\partial y}\Big)^2}\,.\end{aligned}$$

We observe that by trivial algebra, this last expression is precisely $\sqrt{EG-F^2}$.

PROBLEM 2.5. Show that the stars in the Ptolemaic frame of reference (see Section 12 of Chapter 1) are *not* moving along geodesics in the non-Euclidean geometry of the surface that was used to reconcile the motion with special relativity. (Therefore, they must have some acceleration tangential to the surface. In relativistic language, the space-time metric cannot be the flat one of the special theory of relativity.)

Solution: We can show that the level curves on any surface of revolution $z=f(r)$ are *not* geodesics. In fact, given that

$$F(r,r',\theta') = \sqrt{\big(1+\big(f'(r)\big)^2\big)(r')^2+r^2(\theta')^2}\,,$$

the Euler equations say, when arc length s is the parameter,

$$\begin{aligned} r(\theta')^2 &= \big(1+\big(f'(r)\big)^2\big)r''+f'(r)f''(r)(r')^2\\ \theta' &= \frac{k}{r^2}\,,\end{aligned}$$

for some constant k.

It follows from the first of these that, if r is constant, then either θ is constant or $r\equiv 0$. Only the second of these is the equation of a level curve, and it is a trivial (constant) level curve that goes nowhere.

PROBLEM 2.6. Show how to state the vibrating string problem in Newtonian language, using $F=ma$.

Solution: The force acts vertically, and so it is just the vertical component of the tension: $F=-T\sin\alpha$, where $\alpha=\arctan(\partial y/\partial x)$ is the inclination of the curve. Since we are assuming the slope is small enough to allow the equation $\sin\alpha\approx\tan\alpha$

to be used as exact, this gives the forces at the two ends of the small segment we are considering:

$$\begin{aligned} F_x &= -T\frac{\partial y(x,t)}{\partial x} \\ F_{x+dx} &= T\frac{\partial y(x+dx,t)}{\partial x}\,. \end{aligned}$$

(The second of these lacks the negative sign, since it is the force on the portion of the string to the right of this segment, while the other is the force on the portion to the left of it.)

The total vertical component of the force is thus

$$F_x + F_{x+dx} = T\left(\frac{\partial y(x+dx,t)}{\partial x} - \frac{\partial y(x,t)}{\partial x}\right) = T\frac{\partial^2 y}{\partial x^2}\,dx\,.$$

Since

$$ma = \rho\frac{\partial^2 y(x,t)}{\partial t^2}\,dx\,,$$

Newton's second law now gives the standard equation.

Problem 2.7. Show that one solution to the equation of the vibrating string, given by Jean le Rond d'Alembert (1717–1783), is

$$y(x,t) = \frac{f(x+ct)+f(x-ct)}{2}\,,$$

where $y = f(x)$ is the initial configuration of the string at time $t = 0$, extended to all positive and negative values of x as an odd function of period $2L$, and that this solution has initial velocity 0.

That is, the string is stretched into the shape given by this function and suddenly released. This solution shows that the (unique) solution to this initial-value problem when the ends of the string are clamped is the average of the two waves generated by moving the initial configuration right and left with velocity c.

Solution: This is routine. Substituting this value of $y(x,t)$ into the equation makes both sides equal to

$$c^2\frac{f''(x+ct)+f''(x-ct)}{2}\,.$$

It is obvious that $y(x,0) = f(x)$ for $0 \le x \le L$, and that

$$\frac{\partial y(x,t)}{\partial t} = c\frac{f'(x+ct)-f'(x-ct)}{2}$$

becomes zero when $t = 0$.

We note in addition that $y(0,t) = \big(f(ct)+f(-ct)\big)/2 = 0$ for all t, since f is an odd function, and that

$$y(L,t) = \frac{f(L+ct)+f(L-ct)}{2} = \frac{f(ct+L)-f(ct-L)}{2} = 0$$

for all t, since f is an odd function and has period $2L$.

Problem 2.8. Show that another way of solving the same problem, that is, given an initial configuration in the shape of the graph of $y = f(x)$ and initial velocity 0, is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\cos(nct)\,,$$

with the constants c_n chosen so that

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

Make whatever assumptions as to smoothness, differentiability, and convergence of the series you find convenient. This solution was given by Daniel Bernoulli (1700–1782), the son of Johann Bernoulli, a Swiss compatriot of Euler, and a colleague of Euler's for a while at the Russian Academy of Sciences during the 1730s.

Solution: It is a routine computation to show that the function

$$f_n(x,t) = \sin\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi n c t}{L}\right)$$

satisfies the clamping conditions at the endpoints, that is $f_n(0,t) = f_n(L,t)$ for all t, and differentiation shows that each of these functions is a solution of the differential equation.

Thus, if we merely assume that the series can be differentiated termwise, we know that we have a solution. Moreover, the initial velocity is certainly zero, since

$$\frac{\partial y}{\partial t} = -\frac{\pi c}{L}\sum_{n=1}^{\infty} n c_n \sin\left(\frac{\pi n x}{L}\right)\sin\left(\frac{\pi n c t}{L}\right),$$

each term of which is zero when $t = 0$.

Finally, if we use the identity

$$\sin(a)\cos(b) = \frac{1}{2}\big(\sin(a+b) + \sin(a-b)\big),$$

we can write the solution as

$$y(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n(x+ct)}{L}\right) + c_n \sin\left(\frac{\pi n(x-ct)}{L}\right) = \frac{f(x+ct)+f(x-ct)}{2}.$$

This is precisely d'Alembert's solution, assuming the Fourier series of $f(x)$ converges.

APPENDIX 3

Point-Set Topology

Problem 3.1. Prove that $\partial E = \partial(X \setminus E)$.

Solution:

$$\partial E = X \setminus \big(\mathrm{int}\,(E) \cup \mathrm{ext}\,(E)\big) = X \setminus \big(\mathrm{ext}\,(X \setminus E) \cup \mathrm{int}\,(X \setminus E)\big) = \partial(X \setminus E)\,.$$

Problem 3.2. Prove that a subset E of a topological space X is closed if and only if $\partial E \subseteq E$.

Solution: The set E is closed if and only if $X \setminus E$ is open, which is to say $X \setminus E = \mathrm{int}\,(X \setminus E) = \mathrm{ext}\,E$. Then $E = X \setminus (X \setminus E) = X \setminus \mathrm{ext}\,E = \mathrm{int}\,E \cup \partial E$, and so $\partial E \subseteq E$.

Problem 3.3. Prove that $(E^c)^c = E^c$, and that $\varnothing^c = \varnothing$.

Solution: The set E^c is the intersection of all closed sets containing E, since it is the complement of $\mathrm{ext}\,(E)$, which is the union of all open sets disjoint from E. Thus, if a closed set contains E, it also contains E^c. Since any intersection of closed sets is closed, it follows that E^c is closed. In other words, E^c is characterized as the unique *minimal* closed set containing E. Then $(E^c)^c$, the intersection of all closed sets containing E^c, must be just E^c itself, since it is one of the closed sets that contain it, and therefore the smallest one.

Problem 3.4. Prove that $(E \cup F)^c = (E^c \cup F^c)$.

Solution: Any closed set that contains E and F must also contain E^c and F^c, and so $(E \cup F)^c \supseteq (E^c \cup F^c)$. But $E^c \cup F^c$ is a particular closed set containing E and F, and hence contains the minimal closed set that contains them both. That is, $(E \cup F)^c \subseteq (E^c \cup F^c)$

Problem 3.5. Prove that if F_α is a connected set for each $\alpha \in A$ and $F_\alpha \cap F_\beta \neq \varnothing$ for one index $\beta \in A$ and every $\alpha \in A$, then the union of the sets F_α is connected.

Solution: As in all problems involving connectivity, one proves connectedness indirectly, by assuming a disconnection and deriving a contradiction from that assumption. Suppose then that E and F are open sets having the following properties:

(1) $\bigcup_{\alpha \in A} F_\alpha \subseteq E \cup F$;

(2) $E \cap \bigcup_{\alpha \in A} F_\alpha \neq \varnothing \neq F \cap \bigcup_{\alpha \in A} F_\alpha$;

(3) $E \cap F \cap \bigcup_{\alpha \in A} F_\alpha = \varnothing$.

Since $F_\beta \subseteq E \cup F$ and $F_\beta \neq \varnothing$, assume without loss of generality that $F_\beta \cap F \neq \varnothing$, say $y \in F_\beta \cap F$. Since $E \cap \bigcup F_\alpha \neq \varnothing$, let α be an index such that $E \cap F_\alpha \neq \varnothing$, say $x \in E \cap F_\alpha$. Finally, since $F_\alpha \cap F_\beta \neq \varnothing$, let $z \in F_\alpha \cap F_\beta$. Since F_β is connected, it follows that $z \in F$; since F_α is connected, it follows that $z \in E$. But that implies that $z \in E \cap F \cap \bigcup_{\alpha \in A} F_\alpha \neq \varnothing$, contrary to hypothesis.

Therefore, $\bigcup_{\alpha \in A} F_\alpha$ is connected.

PROBLEM 3.6. Prove that if E and F are compact subsets of X, then $E \cup F$ is also.

Solution: Let $\bigcup_{\alpha \in A} U_\alpha$ be a covering of $E \cup F$ by open sets. It is, in particular, a covering of E and a covering of F. There is a finite set of indices $\alpha_1, \ldots, \alpha_m$ such that

$$E \subseteq \bigcup_{i=1}^{m} U_{\alpha_i}$$

and a finite set of indices $\beta_1, \ldots, \beta_n$ such that

$$F \subseteq \bigcup_{j=1}^{n} U_{\beta_j}\,.$$

Then the sets U_{α_i} and U_{β_j}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ form a finite subcovering of $E \cup F$.

PROBLEM 3.7. Let A be a subset of the real line and $-A = \{-x : x \in A\}$. Prove that $-A$ is bounded below (resp. above) if and only if A is bounded above (resp. below) and that the greatest lower bound (resp. least upper bound) of $-A$ is $-b$, where b is the least upper bound (resp. greatest lower bound) of A.

Solution: The basic properties of the negative sign show that if c is an upper (resp. lower) bound of A, then $-c$ is a lower (resp. upper) bound of $-A$, and that if c is the least upper (resp. greatest lower) bound of A, then $-c$ is the greatest lower (resp. least upper) bound of $-A$. (By definition, c being the least upper bound of A means that if a is *any* upper bound of A, then $c \leq a$.)

PROBLEM 3.8. Prove that the L_∞ norm $\|f\|_\infty = \max\{|f(x)| : x \in [a, b]\}$ makes the space of continuous functions on $[a, b]$ into a metric space. (Show that the triangle inequality holds.)

Solution: Compactness and continuity guarantee that $\|f\|_\infty$ is finite and that there actually is a point $c \in [a, b]$ where $|f(c)| = \|f\|_\infty$. Now we have

$$\begin{aligned} \|f + g\|_\infty &= \max\{|f(x) + g(x)|\,;\, x \in [a, b]\} \\ &= |f(c) + g(c)| \text{ for some } c \in [a, b] \\ &\leq |f(c)| + |g(c)| \\ &\leq \|f\|_\infty + \|g\|_\infty\,. \end{aligned}$$

PROBLEM 3.9. Prove that $\partial(E \cap F) \subseteq \partial E \cup \partial F$, and show that in general no stronger statement than this can be made. In particular, it is emphatically not true in general that if $E \subseteq F$, then $\partial E \subseteq \partial F$.

Solution: We note the fundamental facts that $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$ and $\operatorname{int}(A) \cup \operatorname{int}(B) \subseteq \operatorname{int}(A \cup B)$. It is obvious that if $C \subseteq D$, we have $\operatorname{int}(C) \subseteq \operatorname{int}(D)$, and that proves the second of these relations immediately. As for the first relation, we get immediately that $\operatorname{int}(A \cap B) \subseteq \operatorname{int}(A) \cap \operatorname{int}(B)$ from the fact that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. The opposite inclusion is also easy, since if $x \in \operatorname{int}(A) \cap \operatorname{int}(B)$, there are open sets $U \subseteq A$ and $V \subseteq B$ such that $x \in U$ and $x \in V$. Then $U \cap V$ is an open set such that $x \in U \cap V \subseteq A \cap B$, and so $x \in \operatorname{int}(A \cap B)$. A consequence of the second relation is the following set inclusion:

$$\begin{aligned} \operatorname{ext}(A \cap B) &= \operatorname{int}\big(X \setminus (A \cap B)\big) \\ &= \operatorname{int}\big((X \setminus A) \cup (X \setminus B)\big) \\ &\supseteq \operatorname{int}(X \setminus A) \cup \operatorname{int}(X \setminus B) \\ &= \operatorname{ext}(A) \cup \operatorname{ext}(B), \end{aligned}$$

so that $X \setminus \operatorname{ext}(A \cap B) \subseteq X \setminus \big(\operatorname{ext}(A) \cup \operatorname{ext}(B)\big)$. It then follows that

$$\begin{aligned} \partial(E \cap F) &= X \setminus \big(\operatorname{int}(E \cap F) \cup \operatorname{ext}(E \cap F)\big) \\ &= \Big(X \setminus \big(\operatorname{int}(E) \cap \operatorname{int}(F)\big)\Big) \cap \big(X \setminus \operatorname{ext}(E \cap F)\big) \\ &\subseteq \Big(\big(X \setminus \operatorname{int}(E)\big) \cup \big(X \setminus \operatorname{int}(F)\big)\Big) \cap \Big(X \setminus \big(\operatorname{ext}(E) \cup \operatorname{ext}(F)\big)\Big) \\ &= \big(\partial E \cup \operatorname{ext}(E) \cup \partial F \cup \operatorname{ext}(F)\big) \cap \Big(X \setminus \big(\operatorname{ext}(E) \cup \operatorname{ext}(F)\big)\Big) \\ &\subseteq \partial(E) \cup \partial(F). \end{aligned}$$

The set operations in the last step here amount to starting with the two pairs of disjoint sets $\operatorname{ext}(E), \partial(E)$ and $\operatorname{ext}(F), \partial(F)$, taking the union of all four, then excluding both $\operatorname{ext}(E)$ and $\operatorname{ext}(F)$. It is clear that doing so leaves a subset of $\partial(E) \cup \partial(F)$, but perhaps not all of this union, since $\partial(E) \cap \operatorname{ext}(F)$ and $\partial(F) \cap \operatorname{ext}(E)$ are removed.

As an extreme example of strict inequality here, one can take E to be the rational numbers and $F = X$ the real numbers. Then $\partial E = X$ and $\partial F = \varnothing$.

(It is possible to get $\partial(E \cap F) = \partial E \cup \partial F$. For example, if E is the interval $(-2, 2)$ and F the set $(-\infty, -1) \cup (1, \infty)$, we have $E \cap F = (-2, -1) \cup (1, 2)$, and its boundary is $\{-2, -1, 1, 2\}$, which is precisely $\partial E \cup \partial F$.)

Problem 3.10. Prove that $(E \cup F)' = E' \cup F'$, and in particular, if $E \subseteq F$, then $E' \subseteq F'$.

Solution: It seems clear that $E' \cup F'$ is contained in $(E \cup F)'$, since if $x \in E'$, then every deleted neighborhood of x contains a point of E, and hence also a point of $E \cup F$, and similarly if $x \in F'$. We need to show the converse: If $x \in (E \cup F)'$, then either $x \in E'$ or $x \in F'$. To do that, suppose $x \notin E' \cup F'$. Then $x \notin E'$ and $x \notin F'$. Hence, there exists a neighborhood U of x such that $U \setminus \{x\}$ is disjoint from E and a neighborhood V of x such that $V \setminus \{x\}$ is disjoint from F. Then $U \cap V$ is a neighborhood of x such that $(U \cap V) \setminus \{x\}$ is disjoint from $E \cup F$, and so $x \notin (E \cup F)'$.

Problem 3.11. The closure operator ^{c} has the following properties:

(1) $E \subseteq E^c$ for all $E \subseteq X$;
(2) $\varnothing^c = \varnothing$.

(3) $(E \cup F)^c = E^c \cup F^c$.
(4) $(E^c)^c = E^c$ for all $E \subseteq X$.

As it turns out, any operator with these properties defines a topology on X, in which the closed sets are those sets E that are left fixed by the operator, that is, $E^c = E$. Prove this fact.

Solution: We need to show that $\varnothing$ and X are closed sets, that the union of two closed sets is closed, and that any intersection of closed sets is closed. That $\varnothing^c = \varnothing$ is one of the hypotheses of the theorem, and that $X^c = X$ follows from the axiom that $E \subseteq E^c$. As for the union, if $E = E^c$ and $F = F^c$, then $(E \cup F)^c = E^c \cup F^c = E \cup F$.

To prove that these sets are closed under arbitrary intersections, we note that the third property implies that if $E \subseteq F$, then $E^c \subseteq F^c$. Therefore, if $E_\alpha^c = E_\alpha$ for all α in some index set A, then

$$\Bigl(\bigcap_{\alpha \in A} E_\alpha\Bigr)^c \subseteq E_\beta^c = E_\beta$$

for all indices $\beta \in A$, and therefore

$$\Bigl(\bigcap_{\alpha \in A} E_\alpha\Bigr)^c \subseteq \bigcap_{\beta \in A} E_\beta = \bigcap_{\alpha \in A} E_\alpha \subseteq \Bigl(\bigcap_{\alpha \in A} E_\alpha\Bigr)^c,$$

which implies that

$$\Bigl(\bigcap_{\alpha \in A} E_\alpha\Bigr)^c = \bigcap_{\alpha \in A} E_\alpha\,,$$

that is, the set

$$\bigcap_{\alpha \in A} E_\alpha$$

is closed.

Problem 3.12. A good example of a closure operator is the hull-kernel operator on the space of maximal ideals of a ring. For our purposes, the integers will serve as a typical ring, that is, a structure on which addition and multiplication are defined, with the usual properties of associativity, commutativity, and distributivity. (If the ring is an *algebra*, that is, in addition to its structure as a ring, it is a vector space over some field, say the real or complex numbers, an ideal is also required to be closed under scalar multiplication.)

An *ideal* $\mathfrak{A}_1$ in a ring $\mathfrak{A}$ is a subset of $\mathfrak{A}$ that is also a ring under the addition and multiplication (and scalar multiplication, if the ring is an algebra) induced from $\mathfrak{A}$ and is such that if $x \in \mathfrak{A}_1$ and $y \in \mathfrak{A}$, then $xy \in \mathfrak{A}_1$.

Obviously, $\mathfrak{A}$ itself is an ideal, as is the subring consisting of the zero element alone. We shall consider only commutative rings, that is, those in which $xy = yx$ for all x and y. Otherwise we should have to talk about left ideals and right ideals. We shall also assume that the rings we deal with have an identity for multiplication, which we denote 1.

In the case of the ring of integers, if $\mathfrak{A}_1$ is an ideal different from the two just mentioned, then it contains a smallest positive integer a_1, and we claim that $\mathfrak{A}_1$ consists of precisely the multiples of a_1. This is easy to prove, since if a is any integer, there exist integers q and r such that $a = qa_1 + r$ and $0 \le r < a_1$. (That is the division-with-remainder algorithm.) If $a \in \mathfrak{A}_1$, then $r = a - qa_1$ also belongs

to $\mathfrak{A}_1$, and by definition of a_1, we must have $r = 0$. Thus $a = qa_1$, as asserted. We call the set of multiples of a_1, the *principal ideal* generated by a_1, and we denote it (a_1). Notice that if a_1 and a_2 are positive integers such that $(a_2) \subseteq (a_1)$, then $a_2 \in (a_1)$ and hence a_2 is a multiple of a_1. It is easy to see that if a and b are relatively prime, then $(a) \cap (b) = (ab)$.

An ideal $\mathfrak{I}$ different from $\mathfrak{A}$ is *maximal* if the only ideal of which it is a proper subset is $\mathfrak{A}$ itself. In the case of the integers, an ideal (p) is maximal if and only if p is prime. For $(ab) \subseteq (a)$ and $(ab) \neq (a)$ when $b > 1$, because $a \in (a) \setminus (ab)$. And if $\mathfrak{I}$ is an ideal that properly contains a prime ideal (p), then it contains an integer q that is not a multiple of p, and so is relatively prime to p. But then the greatest common divisor of p and q is 1, and this number is of the form $ap + bq$, hence in $\mathfrak{I}$. But obviously the only ideal that contains 1 is the whole ring of integers. (This is true in general, in any ring that has an identity for multiplication.)

The *kernel* of a set of ideals of a ring $\mathfrak{A}$ is the ideal that is their intersection.

In the ring of integers, the kernel of a *finite* set of ideals $(a_1),\ldots,\ (a_n)$ is the ideal (a), where a is the least common multiple of $a_1, \ldots, a_n$. The kernel of any infinite set of distinct ideals in this ring is the set $\{0\}$.

The *hull* of a subset E of $\mathfrak{A}$ is the collection of all maximal ideals that contain E.

The hull is not an ideal; it is a set whose elements are maximal ideals. The entire set of maximal ideas of a ring is called (naturally) the *maximal ideal space* of the ring. In the case of the integers, the maximal ideal space is in one-to-one correspondence with the set of positive prime numbers. In this example, the hull of any set E is the set of all prime divisors of all elements of E. The hull of the kernel of a finite set of ideals $(a_1),\ldots,\ (a_n)$ is the set of prime ideals generated by factors of the least common multiple of $a_1,\ldots,\ a_n$, that is, the set of primes that divide one or more of the numbers $a_1, \ldots, a_n$. The following proposition connects all these algebraic concepts with point-set topology.

The operator ^{c} defined on the set of maximal ideals of a ring $\mathfrak{A}$ by specifying that $E^c = \mathrm{hull}\big(\mathrm{kernel}\,(E)\big)$, is a closure operator.

Proof. It is obvious that $E \subseteq E^c$, since each maximal ideal in a collection contains the intersection of all the maximal ideals in the collection. By convention, if $E = \varnothing$ is the empty set of maximal ideals, its kernel—the intersection of the (non-existent) ideals in E—is the whole ring $\mathfrak{A}$. Since there are no maximal ideals containing $\mathfrak{A}$, it follows that the hull of $\mathfrak{A}$ is empty, so that $\varnothing^c = \varnothing$.

Next, if $\mathfrak{A}_1 \in E^c$, then $\mathfrak{A}_1$ contains the intersection of all the ideals in E. But that means that if we adjoin $\mathfrak{A}_1$ to E, the intersection does not get any smaller, since it was already contained in $\mathfrak{A}_1$. Obviously, it does not get any larger either, so that the kernel of E is also the kernel of E^c. Therefore E^c, the hull of the kernel of E, is also the hull of the kernel of E^c, and that means $(E^c)^c = E^c$.

Now the kernel of a union of two sets E and F of maximal ideals is obviously contained in the kernel of E and in the kernel of F, and hence its hull contains the hull of the kernel of E and the hull of the kernel of F, that is, $(E \cup F)^c \supseteq E^c \cup F^c$. When $E \subseteq F$, this relation implies $F^c \supseteq E^c \cup F^c$, so that $E^c \subseteq F^c$.

Conversely, if $\mathfrak{I}$ is a maximal ideal and $\mathfrak{I} \notin E^c \cup F^c$, then $\mathfrak{I}$ does not contain the kernel of E and it does not contain the kernel of F. That is, there is a point $x \in \ker(E) \setminus \mathfrak{I}$ and a point $y \in \ker(F) \setminus \mathfrak{I}$. Since $\ker(E)$ and $\ker(F)$ are ideals, the

point $z = yx$ belongs to $\ker(E) \cap \ker(F) = \ker(E \cup F)$. Then, either $\mathfrak{I} \notin (E \cup F)^c$, or $z \in \mathfrak{I}$. The former is what we would like to show, since we want to show that $(E \cup F)^c = E^c \cup F^c$. So, suppose $z \in \mathfrak{I}$. Now $\mathfrak{I}$ is a *maximal* ideal that does not contain x. That means that the smallest ideal containing $\mathfrak{I}$ and having x as one of its points is the whole ring. This ideal can be described as the set of all points $w + xu$, as w ranges over $\mathfrak{I}$ and u ranges over the whole ring. In particular $1 - xu \in \mathfrak{I}$ for some u, and therefore $y - yxu \in \mathfrak{I}$. But since $yxu = zu \in \mathfrak{I}$, this means $y \in \mathfrak{I}$, contradicting the choice of y. Thus, we do not have $z \in \mathfrak{I}$, and we are done. $\square$

What are the closed sets in the maximal ideal space of the integers? Describe the topology generated by this closure operator.

Solution: Since the kernel of an infinite set of prime ideals is $\{0\}$, its hull is the entire set of prime ideals. Thus, the smallest closed set containing an infinite set of prime ideals is the entire maximal ideal space. An infinite set is therefore closed only if it is the whole maximal ideal space.

A finite set of prime ideals is closed, since it is the hull of its kernel. (The kernel of the set of prime ideals $\{(p_1), \ldots, (p_n)\}$ is the ideal $(p_1 p_2 \cdots p_n)$, and its hull is once again the set $\{(p_1), \ldots, (p_n)\}$.)

Thus the closed sets in this topology are the whole maximal ideal space and the finite sets, so that the open sets are the empty set $\varnothing$ and the sets whose complements are finite. It is the co-finite topology discussed in the text.

Problem 3.13. Here is an amusing example of the use of a topology to prove a seemingly unrelated fact. It is due to Hillel Furstenberg (see his article "On the infinitude of primes" in the 1955 *American Mathematical Monthly*, **62**, p. 353). Let $X = \mathbb{Z}$ be the set of integers (positive, negative, and zero). For each nonnegative integer a and positive integer $d > a$, define the set

$$B(a,d) = \{a + nd : \, n \in \mathbb{Z}\} .$$

In other words, $B(a,d)$ is an arithmetic sequence with difference d; putting the matter yet another way, it is the set of all integers that are equal to a modulo d. It is clear that if $B(a_1,d_1) \cap B(a_2,d_2)$ is non-empty, and b is the smallest non-negative integer it contains, then

$$B(a_1,d_1) \cap B(a_2,d_2) = B(b,d) ,$$

where d is the least common multiple of d_1 and d_2. This is the one condition that needs to be met in order for the sets $B(a,d)$ to form a base for a topology $\mathfrak{T}$ on $\mathbb{Z}$.

Now, we make two observations about this topology:

(1) The basic open sets are both open and closed, since

$$X \setminus B(a,d) = \bigcup_{0 \le b < d \, , \, b \ne a} B(b,d) .$$

(2) A non-empty open set in this topology is infinite (since it contains an infinite set $B(a,d)$ for some a and d).

That being given, consider the union of all prime ideals:

$$P = \cup_{p \text{ prime}} B(0,p) .$$

Show that if the set of prime numbers were finite, then $\mathbb{Z} \setminus P$ would be a finite, non-empty open set in this topology. Hence, *there are infinitely many prime numbers!* (Just to be clear: The number 1 is *not* a prime number.)

Solution: If there were only finitely many prime numbers, the union of all the prime ideals would be a finite union of closed sets (since each prime ideal is one of the basic open sets, which are also closed), and hence would be closed. Its complement is the two-point set $\{-1, +1\}$, which would then have to be a finite, non-empty open set, which does not exist in this topology. Thus, there have to be infinitely many prime numbers.

PROBLEM 3.14. A point x in a topological space X is *isolated* if the singleton set $\{x\}$ is an open set. Prove that a subset E of X that contains an isolated point x of X is of second category on X.

Solution: If E is a countable union of sets E_n, then $x \in E_n$ for some n, and then the interior of the closure of E_n contains the point x. Therefore E_n fails to be nowhere-dense, and so E is not a countable union of nowhere-dense sets. It is therefore of second category.

PROBLEM 3.15. A topological space X is *separable* if there is a (finite or) countable set $E = \{x_1, \ldots, x_n, \ldots,\}$ such that $E^c = X$. Such a set E is said to be *dense in* X. Prove that a second-countable topological space is separable and first-countable.

Solution: If $\{U_n\}_{n=1}^{\infty}$ is a countable base (of non-empty sets) for the topology of a space X, then for each $x \in X$, the sets $\{U_n : x \in U_n\}$ form a countable base of neighborhoods of x. If $z_n \in U_n$ for each n, then $\{z_n\}_{n=1}^{\infty}$ is a countable dense subset of X.

PROBLEM 3.16. Exhibit a countable dense subset of $\mathbb{R}^n$.

Solution: Fix any non-zero real number a, and let E be the set of all points $a\boldsymbol{r} = (ar^1, \ldots, ar^n)$, where $r^1, \ldots, r^n$ are rational numbers.

PROBLEM 3.17. Show that a compact Hausdorff space X is rich in continuous real-valued functions, in the sense that if A and B are any two disjoint closed subsets of X, there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. (*Outline of the proof:* First show that if C_0 is a closed subset, U_1 an open subset, and $C_0 \subseteq U_1$, then there exists an open subset $U_{1/2}$ such that $C_0 \subseteq U_{1/2} \subseteq U_{1/2}^c \subseteq U_1$. Then start with $C_0 = A$, $U_1 = X \setminus B$, and define by induction on n sets $U_{k/2^n}$ for $0 < k < 2^n$ such that $U_{(k-1)/2^n}^c \subseteq U_{k/2^n}$. For example, if k is even, then $U_{k/2^n} = U_{l/2^{n-1}}$, so that $U_{k/2^n}$ is already defined at the previous stage as an open set containing the closed set $U_{(l-1)/2^{n-1}}^c$, where $l = k/2$. And similarly if k is odd, then $U_{(k-1)/2^n}^c = U_{l/2^{n-1}}^c$, so that $U_{(k-1)/2^n}^c$ is already defined at the previous stage as a closed set contained in the open set $U_{(l+1)/2^n}$. In either case, you can apply the construction just shown to get the required sequence. Then for each $x \in X$, let $f(x) = \inf\{r : x \in U_r\}$, where r ranges over the binary rationals in $[0, 1]$. To show that f is continuous, show that the inverse image of an interval $[0, b)$ is open, and that the inverse image of $(a, 1]$ is open. Since these sets and their finite intersections form a base of the topology of $[0, 1]$ it follows that f is continuous.)

Solution: Since the space X is assumed to be a Hausdorff space, and is compact, its closed subsets are the same as its compact subsets. We have already shown in the text how two disjoint compact sets E and F can be separated by a pair of disjoint open sets U and V, that is, $E \subseteq U$, $F \subseteq V$, and $U \cap V = \varnothing$. It follows in particular that $F \subseteq \mathrm{ext}\,(U)$, so that $U^c \cap F = \varnothing$, that is, $U^c \subseteq X \setminus F$. Applying this result with $E = A = C_0$, $F = B$, $X \setminus F = X \setminus B = U_1$, we can take $U_{1/2} = U$. We then have, as required, $C_0 \subseteq U_{1/2} \subseteq U_{1/2}^c \subseteq U_1$. We can then start over with the new closed-open pairs $(C_0, U_{1/2})$ and $(C_{1/2}, U_1)$, where $C_{1/2} = U_{1/2}^c$. In this way, we get an open set U_r corresponding to each binary rational number $r = k/2^n$, defined by induction on n, and having the property that $U_r^c \subseteq U_s$ if $r < s$.

Still following the suggested outline of the proof, we define a function $f : X \to [0, 1]$ by specifying that

$$f(x) = \begin{cases} 1\,, & \text{if } x \notin \cup_r U_r\,, \\ \inf\{r \,:\, x \in U_r\}\,, & \text{if } x \in U_r \text{ for some } r\,. \end{cases}$$

We claim that f is a continuous function. To that end, following the suggested proof, we consider $f^{-1}\big([0, a)\big)$ and $f^{-1}\big((b, 1]\big)$ for $0 < a \leq 1$ and $0 \leq b < 1$. (The inverse image is obviously the empty set if $a = 0$ or $b = 1$.)

To say $x \in f^{-1}\big([0, a)\big)$, where $0 < a \leq 1$, is equivalent to saying that there exists $r < a$ such that $x \in U_r$. Thus

$$f^{-1}\big([0, a)\big) = \bigcup_{r<a} U_r\,,$$

which is obviously an open set.

To say $x \in f^{-1}\big((b, 1]\big)$, where $0 \leq b < 1$ is to say that $x \notin U_s$ for some $s > b$. If U_s is such that $x \notin U_s$, and $s > b$, there exists a binary rational number r satisfying $b < r < s$, and then $U_r^c \subseteq U_s$, and hence $x \notin U_r^c$. Thus

$$f^{-1}\big((b, 1]\big) = \bigcup_{r>b} \big(X \setminus U_r^c\big)\,,$$

which is obviously the union of complements of closed sets and therefore open.

Problem 3.18. Let X be a compact space and Y a Hausdorff space, and let $f : X \to Y$ be a one-to-one continuous mapping of X onto Y. Prove that the inverse mapping $f^{-1} : Y \to X$ is continuous. It follows easily that X is also a Hausdorff space, Y is a compact space, and a subset V of Y is open if and only if $U = f^{-1}(V)$ is an open subset of X. The mapping f (or f^{-1}) is called a *homeomorphism*. It follows that, given a compact Hausdorff topology on any set X, the space X is not a Hausdorff space in any strictly weaker topology and not a compact space in any strictly stronger one. There can be two different compact Hausdorff topologies on a given set, but neither can be stronger than the other; that is, each contains some open sets not in the other. Compact (more generally, locally compact) Hausdorff spaces satisfy the “Goldilocks principle” when choosing topologies. (But it is not always possible to get one that meets our needs; infinite-dimensional real or complex normed vector spaces, for example, are not locally compact.)

Solution: Since X is compact, Y is also compact, being the continuous image of a compact space. If x_1 and x_2 are distinct points of X, then their images $f(x_1)$ and $f(x_2)$ are distinct points of Y, since f is one-to-one. Then there are disjoint open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$ and $f(x_2) \in V_2$. If we define

$U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$, then U_1 and U_2 are disjoint open sets in X and $x_1 \in U_1$, $x_2 \in U_2$. Hence X is a Hausdorff space, as asserted.

It now follows that the closed sets and the compact sets in X are the same and the closed sets and the compact sets in Y are the same. Since the image of a compact set is compact, while the inverse image of a closed set is closed, we see that $C \subseteq X$ is closed if and only if $f(C) \subseteq Y$ is closed, and that implies that $f^{-1} : Y \to X$ is continuous.

PROBLEM 3.19. The space ℓ^1 is defined as the set of all sequences $A = \{a_n\}_{n=0}^{\infty}$ of complex numbers (note that the index begins with 0) such that

$$\sum_{n=0}^{\infty} |a_n| < \infty$$

with addition and scalar multiplication defined termwise and multiplication defined as *convolution*:

$$A * B = C\,,$$

where, with obvious notation,

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}\,.$$

It is not difficult to show that $C \in \ell^1$ if $A \in \ell^1$ and $B \in \ell^1$. With each $A \in \ell^1$ we can associate a function $\widehat{A}(z)$ that is continuous in the closed unit disk $|z| \le 1$ and analytic in the open disk where $|z| < 1$ by the rule

$$\widehat{A}(z) = \sum_{n=0}^{\infty} a_n z^n\,.$$

We observe that A can be recovered from $\widehat{A}(z)$, since $a_n = \widehat{A}^{(n)}(0)/n!$. Thus the mapping $A \mapsto \widehat{A}(z)$ is one-to-one. This rule therefore defines an isomorphism from ℓ^1 *into* (not *onto*) the space of continuous functions in the closed disk that are analytic in its interior. (Not every continuous function of period 2π has an absolutely convergent Fourier series, even if all its Fourier coefficients of negative index are equal to zero. Thus, the mapping is not "onto.") Finally, we remark that the linear mapping $A \mapsto \widehat{A}(z)$ is continuous in the sense that $\|\widehat{A}\|_\infty = \sup|\widehat{A}(z)| \le \sum |a_n| = \|A\|_1$, where $\|A\|_1$ is defined as follows: If $A = (a_0, a_1, \ldots, a_n, \ldots)$, then

$$\|A\|_1 = \sum_{n=0}^{\infty} |a_n|\,.$$

It is easy to show that ℓ^1 is a complete metric space. For, let $\{C_k\}_{k=1}^{\infty}$ be a Cauchy sequence in this metric, where $C_k = (c_{k0}, c_{k1}, \ldots, c_{kn}, \ldots)$. This means that for every $\varepsilon > 0$ there is an integer N such that $\|C_k - C_l\|_1 < \varepsilon$ if $k > N$ and $l > N$. Then for each n, it follows that $|c_{kn} - c_{ln}| < \varepsilon$ also, that is, $\{c_{kn}\}_{k=1}^{\infty}$ is a Cauchy sequence of complex numbers. Since the complex numbers are a complete metric space, there is a complex number c_n such that $c_{kn} \to c_n$ as $k \to \infty$. If $C = (c_0, c_1, \ldots, c_n, \ldots)$, it is then easy to show that $\|C_k - C\|_1$ tends to 0, that is, $C_k \to C$. Thus, ℓ^1 is complete. Let the sequence $\{C_k\}_{k=0}^{\infty} = \big\{\{c_{kn}\}_{n=0}^{\infty}\big\}_{k=0}^{\infty}$ (*not*

a Cauchy sequence!), be given by

$$c_{kn} = \begin{cases} 1 & \text{if } n = k\,, \\ 0 & \text{if } n \neq k\,. \end{cases}$$

Show that the set $\{C_0, C_1, \ldots, C_k, \ldots\}$ is linearly independent and that every element $C \in \ell^1$ can be approximated with arbitrary precision by some finite linear combination of these elements. Thus they are a *geometric* basis of ℓ^1, and its geometric dimension is therefore countably infinite.

Now let $\{C_\alpha\}_{\alpha \in A}$ be a maximal linearly independent set, that is, one for which every element $C \in \ell^1$ is *equal* to some finite linear combination of these elements: $C = t_1 C_{\alpha_1} + \cdots + t_r C_{\alpha_r}$. This set is an *algebraic* basis, and the cardinality of A is the algebraic dimension of the space. Show that the cardinality of the index set A is that of the continuum. *Hint:* Assume that this cardinality is a countable infinity, that is, the maximal linearly independent set just named is $\{C_{\alpha_1}, \ldots, C_{\alpha_n}, \ldots\}$. Show that the finite-dimensional subspace spanned by the first n of these elements is nowhere-dense in ℓ^1. Then invoke the Baire category theorem, since ℓ^1 is the union of these subspaces.

Solution: Let us first complete the verification that ℓ^1 is a complete metric space. We have shown how to find a natural candidate $C = (c_0, c_1, \ldots,)$ for the limit of a Cauchy sequence $\{C_k\}_{k=0}^{\infty}$, namely the sequence C given by

$$c_n = \lim_{k \to \infty} c_{kn}\,.$$

(That is, we showed, using the completeness of the complex numbers, that this limit exists.) It remains to be shown that $C \in \ell^1$ and that $\|C_k - C\|_1 \to 0$. That is,

$$\sum_{n=0}^{\infty} |c_n| < +\infty$$

and

$$\lim_{k \to \infty} \sum_{n=0}^{\infty} |c_{kn} - c_n| = 0\,.$$

To show the first statement, let $\varepsilon > 0$ and fix an integer k_0 such that

$$\sum_{n=0}^{\infty} |c_{kn} - c_{ln}| < \frac{\varepsilon}{2}$$

for all $k \geq k_0$, $l \geq k_0$.

Then fix an integer n_0 such that

$$\sum_{n=n_0}^{\infty} |c_{k_0 n}| < \frac{\varepsilon}{2}\,.$$

If $q > p \geq n_0$, we have

$$\begin{aligned} \sum_{n=p}^{q} |c_n| &\leq \sum_{n=p}^{q} |c_{k_0 n}| + \sum_{n=p}^{q} \big||c_{k_0 n}| - |c_n|\big| \\ &< \frac{\varepsilon}{2} + \sum_{n=p}^{q} |c_{k_0 n} - c_n| \\ &= \frac{\varepsilon}{2} + \lim_{l\to\infty} \sum_{n=p}^{q} |c_{k_0 n} - c_{ln}| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,. \end{aligned}$$

It follows that the series $\sum |c_n|$ is a Cauchy series, that is, its sequence of partial sums is a Cauchy sequence of real numbers and therefore converges to a finite limit. Thus, $C \in \ell^1$, as claimed.

As for the second claim, again let $\varepsilon > 0$, and choose k_0 and n_0 as above. Since p and q were arbitrary integers not smaller than n_0, the argument just given shows that

$$\sum_{n=n_0}^{\infty} |c_n| \leq \varepsilon\,,$$

and the same argument with C_l in place of C shows that

$$\sum_{n=n_0}^{\infty} |c_{ln}| \leq \varepsilon$$

if $l \geq k_0$. Then, if $l \geq k_0$, we have

$$\|C_l - C\|_1 \leq \sum_{n=0}^{n_0-1} |c_{ln} - c_n| + \sum_{n=n_0}^{\infty} |c_{ln}| + \sum_{n=n_0}^{\infty} |c_n| \leq 2\varepsilon + \sum_{n=0}^{n_0-1} |c_{ln} - c_n|\,.$$

The last term is smaller than ε for all sufficiently large values of l, and therefore $\|C_l - C\|_1 \to 0$, as claimed. This finishes the proof that ℓ^1 is a complete metric space.

It is now very easy to show that for the particular sequence C_k, where $c_{kn} = \delta_{kn}$ (the Kronecker δ whose value is 1 if $k = n$ and 0 if $k \neq n$), we have, for any $C \in \ell^1$,

$$C = \lim_{N\to\infty} \sum_{n=0}^{N} c_n C_n\,.$$

This amounts to the obvious fact that

$$\left\| C - \sum_{n=0}^{N} c_n C_n \right\|_1 = \sum_{n=N+1}^{\infty} |c_n|\,,$$

which tends to zero by definition of the fact that $C \in \ell^1$. Thus, ℓ^1 has a countable *geometric* basis.

Now a finite-dimensional subspace of ℓ^1 is closed, as we can easily see. Indeed, let $\mathbb{V}$ be a finite-dimensional subspace of ℓ^1, and let $\{C_1, \ldots, C_r\}$ be a finite, linearly independent set of elements forming a basis of $\mathbb{V}$. The subspace $\mathbb{V}$ inherits the metric $\|C\|_1$ from ℓ^1 and so is a normed vector space. We claim that it is also complete,

and hence must be closed as a subset of ℓ^1. To prove that, we consider the linear transformation $T : \mathbb{C}^n \to \mathbb{V}$ given by

$$T(\boldsymbol{z}) = T(z_1, \dots, z_n) = z_1 C_1 + \cdots + z_n C_n\,.$$

This mapping is obviously a one-to-one mapping of $\mathbb{C}^n$ onto $\mathbb{V}$ and as such has an inverse T^{-1} given by

$$T^{-1}(z_1 C_1 + \cdots + z_n C_n) = (z_1, \dots, z_n)\,.$$

Regarded as a function from one topological (metric) space ($\mathbb{C}^n$) into another ($\mathbb{V}$), T is continuous, since

$$\|T(\boldsymbol{z})\|_1 = \|z_1 C_1 + \cdots + z_j C_n\|_1 \le |z_1|\|C_1\|_1 + \cdots + |z_n|\|C_n\|_1 \le \\ \bigl(|z_1|^2 + \cdots + |z_n|^2\bigr)^{\frac{1}{2}} \bigl(\|C_1\|_1^2 + \cdots + \|C_n\|_1^2\bigr)^{\frac{1}{2}} = K|\boldsymbol{z}|_2\,,$$

where $|\boldsymbol{z}|_2 = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ is the standard (Euclidean) metric on the n-dimensional complex vector space $\mathbb{C}^n$ (properly known as *n-dimensional unitary space* and isometric as a metric space to the real vector space $\mathbb{R}^{2n}$).

It follows from this inequality and linearity that $\|T(\boldsymbol{z}) - T(\boldsymbol{w})\|_1 \le K\|\boldsymbol{z} - \boldsymbol{w}\|_2$. Thus, given $\varepsilon > 0$, take $\delta = \varepsilon/K$, and you get $\|T(\boldsymbol{z}) - T(\boldsymbol{w})\|_1 < \varepsilon$ whenever $|\boldsymbol{z} - \boldsymbol{w}|_2 < \delta$. The mapping T is not only continuous, it is uniformly continuous, in that this inequality is uniform over all of $\mathbb{C}^n$: As long as $\boldsymbol{z}$ and $\boldsymbol{w}$ are within δ of each other, it does not matter where in $\mathbb{C}^n$ they are. Now the inequality $\bigl|\|T(\boldsymbol{z})\|_1 - \|T(\boldsymbol{w})\|_1\bigr| \le \|T(\boldsymbol{z}) - T(\boldsymbol{w})\|_1$ implies that the mapping $|T| : \mathbb{C}^n \to [0, \infty)$ given by $|T|(\boldsymbol{z}) = \|T(\boldsymbol{z})\|_1$ is also continuous. As such, on the compact set[1] $\mathbb{S} = \{\boldsymbol{z} \in \mathbb{C}^n : |\boldsymbol{z}|_2 = 1\}$, the mapping $|T|$ assumes its minimum value, which is not zero. That is, there is a point $\boldsymbol{z}_0 \in S$ such that $\|T(\boldsymbol{z})\|_1 \ge \|T(\boldsymbol{z}_0)\|_1 = k > 0$ for all $\boldsymbol{z}$ satisfying $|\boldsymbol{z}|_2 = 1$. It follows that

$$|T^{-1}(C)|_2 \le \frac{1}{k}\|C\|_1$$

for all $C \in \mathbb{V}$.

Indeed, the vector

$$\boldsymbol{z} = \frac{1}{|T^{-1}(C)|_2} T^{-1}(C)$$

belongs to $\mathbb{S}$, and so

$$k \le \|T(\boldsymbol{z})\|_1 = \frac{1}{|T^{-1}(C)|_2}\|C\|_1\,,$$

Thus, T^{-1} is also continuous, and we have the inequalities

$$k|T^{-1}(C) - T^{-1}(D)|_2 \le \|C - D\|_1 \le K|T^{-1}(C) - T^{-1}(D)|_2\,.$$

It follows from these inequalities that if $\{D_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{V}$, then $\{T^{-1}(D_n)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}^n$, and hence converges to some point $\boldsymbol{w} \in \mathbb{C}^n$. But then, the same inequalities imply that D_n converges to $T(\boldsymbol{w})$, which belongs to $\mathbb{V}$. Thus every Cauchy sequence in $\mathbb{V}$, which, as shown above, converges to a point of ℓ^1, actually has its limit in $\mathbb{V}$. That means that $\mathbb{V}$ contains all of its points of accumulation and is therefore a closed subset of ℓ^1.

We now claim that a finite-dimensional subspace of ℓ^1 (which we now know to be closed) has empty interior in ℓ^1. To see this, let $\mathbb{V}$ be a finite-dimensional

[1] Regarded as a subset of $\mathbb{R}^{2n}$, $\mathbb{S}$ is just the unit sphere that we have called $\mathbb{S}^{2n-1}(1)$ elsewhere in this book.

subspace. Then there exists a vector $C_0 \in \ell^1 \setminus \mathbb{V}$. If U is any neighborhood of a point $C_1 \in \mathbb{V}$, then U contains a ball of radius $r > 0$ about C_1. But this ball contains $C_1 + tC_0$ if $t = r/(2\|C_0\|_1)$, and this vector is not in $\mathbb{V}$, since if it were, $C_0 = \big((C_1 + tC_0) - C_1\big)/t$ would also be in $\mathbb{V}$. Thus, $\mathbb{V}$ contains no non-empty open sets and hence is a nowhere-dense closed subset of ℓ^1.

It now follows that there is no countable *algebraic* basis of ℓ^1. For, if $\{C_1, C_2, \dots\}$ were such a basis, letting $\mathbb{V}_n$ be the subspace of ℓ^1 spanned by $\{C_1, \dots, C_n\}$, we would have

$$\ell^1 = \bigcup_{n=1}^{\infty} \mathbb{V}_n \,,$$

(since by definition of an algebraic basis, every element of ℓ^1 would be a *finite* linear combination of the basis vectors and hence in some $\mathbb{V}_n$), and thus the complete metric space ℓ^1 would be a countable union of nowhere-dense sets, contradicting the Baire category theorem.

Problem 3.20. As mentioned in the preceding problem, if C is any element of ℓ^1 and z any complex number such that $|z| \le 1$, the series

$$\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots$$

converges absolutely and uniformly over all z satisfying the stated inequality (that is, z lying in the closed disk of radius 1 about 0 in the complex plane). Thus, this series defines a function $\widehat{C}(z)$ in that disk that is continuous in the closed disk and analytic—equal to the sum of its Maclaurin series—in the open disk. As a function of C, it turns out to be analogous to the Laplace transform, which is defined (assuming suitable convergence of the integral) as

$$L(\varphi)(z) = \int_0^{\infty} \varphi(t) e^{-tz}\, dt \,.$$

The analogy pairs $\sum$ with $\int$, the sequence C with the function φ, and z^n with $e^{-tz}\, dt$.

Under the convolution operation, the Banach algebra (complete normed algebra) ℓ^1 has a multiplicative identity, namely the sequence $I = (1, 0, 0, \dots)$, and $\widehat{I}(z) \equiv 1$. Formally, if $c_0 \ne 0$, one can solve the infinite system of equations that expresses the equality $C * D = I$ as follows:

$$\begin{aligned}
c_0 d_0 &= 1\,,\\
c_0 d_1 + c_1 d_0 &= 0\,,\\
c_0 d_2 + c_1 d_1 + c_2 d_0 &= 0\,,\\
\vdots \quad &\vdots \quad \vdots,\\
c_0 d_n + c_1 d_{n-1} + \cdots + c_{n-1} d_1 + c_n d_0 &= 0\,,
\end{aligned}$$

getting a sequence $D = (d_0, d_1, \dots, d_n, \dots)$ given by

$$\begin{aligned}
d_0 &= \frac{1}{c_0}\,, \\
d_1 &= -\frac{c_1 d_0}{c_0} = -\frac{c_1}{c_0^2}\,, \\
d_2 &= -\frac{c_2 d_0 + c_1 d_1}{c_0} = -\Bigl(\frac{c_2}{c_0^2} + \frac{c_1^2}{c_0^3}\Bigr),, \\
d_3 &= \frac{c_3 d_0 + c_2 d_1 + c_1 d_2}{c_0} = -\Bigl(\frac{c_3}{c_0^2} + \frac{c_2 c_1 + c_1 c_2}{c_0^3} + \frac{c_1^3}{c_0^4}\Bigr)\,,
\end{aligned}$$

and so on. It is an interesting question whether D belongs to ℓ^1. Show that in general it does not by considering $C = (1, -2, 0, 0, \dots)$.

Solution: It is easy to compute by induction that $d_n = (-2)^n$, which means the radius of convergence of the series is $1/2$. Alternatively, we see that $\widehat{C}(z) = 1 + 2z$, and therefore

$$\frac{1}{\widehat{C}(z)} = 1 - (2z) + (2z)^2 - (2z)^3 + \cdots\,,$$

and this geometric series converges only for $|2z| < 1$. In short, $d_n = (-2)^n$, and so $\sum |d_n| = \infty$.

Problem 3.21. For each fixed z_0 in the closed disk, the set of all $C \in \ell^1$ such that $\widehat{C}(z_0) = 0$ is a maximal ideal $\mathfrak{I}_{z_0}$. That it is an ideal is obvious. That it is maximal follows from the expression $C = \widehat{C}(z_0)(1, 0, 0, \dots) + \bigl(c_0 - \widehat{C}(z_0), c_1, c_2, \dots, c_n, \dots\bigr) = cI + C_1$, where c is the scalar constant $\widehat{C}(z_0)$ and C_1 is such that $\widehat{C}_1(z_0) = 0$, that is, $C_1 \in \mathfrak{I}_{z_0}$. It follows that the ideal $\mathfrak{I}_{z_0}$ has codimension 1 and so is maximal.

Thus, we have achieved a faithful representation of the algebra ℓ^1 as an algebra of functions on (at least part of) its maximal ideal space. (We have not shown that every maximal ideal in ℓ^1 is $\mathfrak{I}_{z_0}$ for some z_0, nor that the hull-kernel topology on the maximal ideal space is the ordinary metric topology of the closed unit disk, although both of these statements are true.)

Show that if $\widehat{C}(z)$ has no zeros in the closed disk, then the sequence D defined in the preceding problem generates a function $\widehat{D}(z) = 1/\widehat{C}(z)$ that is analytic in the open disk, and that this function is continuous on the boundary.

Show that if $d_n \geq 0$ for all large n, and $\widehat{C}(z)$ has no zeros in the closed disk, then $D \in \ell^1$.

This result is a special case of the *Wiener Tauberian theorem*, named after the American mathematician Norbert Wiener (1894–1964).

Solution: The function $1/\widehat{C}(z)$ is analytic in the open disk and continuous in the closed disk, and it follows from the equation $\widehat{C}(z)\bigl(1/\widehat{C}(z)\bigr) \equiv 1$ that $1/\widehat{C}(z)$ has the Maclaurin coefficients d_n computed above. The Maclaurin series therefore converges for $|z| < 1$. Now

$$\lim_{r\uparrow 1} \sum_{n=0}^{\infty} d_n r^n = \lim_{r\uparrow 1} \widehat{D}(r) = \widehat{D}(1)$$

due to the fact that $\widehat{D}$ is continuous on the closed disk, thus the series

$$\sum_{n=0}^{\infty} d_n$$

is Abel-summable with sum $\widehat{D}(1)$. But a series whose terms are of constant sign from some point on actually converges if it is Abel-summable. Hence it follows that $D \in \ell^1$.

APPENDIX 4

Manifolds

PROBLEM 4.1. Verify that the chart mappings $\psi_{11}\circ\psi_{22}^{-1}$, $\psi_N\circ\psi_{11}^{-1}$ and $\psi_{11}\circ\psi_N^{-1}$ that define $\mathbb{S}^n(r)$ as a manifold are analytic. It follows that $\mathbb{S}^n(r)$ is an analytic (C^ω) manifold.

Solution: First we have

$$\psi_{22}^{-1}(x^1,\dots,x^n) = \big(x^1, -\sqrt{r^2-(x^1)^2-\cdots-(x^n)^2}, x^2,\dots,x^n\big)\,,$$

from which it follows that

$$\psi_{11}\circ\psi_{22}^{-1}(x^1,\dots,x^n) = \big(-\sqrt{r^2-(x^1)^2-\cdots-(x^n)^2}, x^2,\dots,x^n\big)\,,$$

and this function is analytic on its domain, which is the set where $(x^1)^2+\cdots+(x^n)^2<r^2$ and $x^1>0$, since the square root function is analytic on the set of positive real numbers.

Next, since

$$\psi_{11}^{-1}(x^1,\dots,x^n) = \big(\sqrt{r^2-(x^1)^2-\cdots-(x^n)^2}, x^1,\dots,x^n\big)\,,$$

it follows that

$$\psi_N\circ\psi_{11}^{-1}(x^1,\dots,x^n) = \left(\frac{r\sqrt{r^2-(x^1)^2-\cdots-(x^n)^2}}{r-x^n}, \frac{rx^1}{r-x^n},\dots,\frac{rx^{n-1}}{r-x^n}\right).$$

Again, this function is analytic on its domain, which is the set where $(x^1)^2+\cdots+(x^n)^2<r^2$. (This inequality implies that $x^n<r$.)

Finally, we have

$$\psi_{11}\circ\psi_N^{-1}(x^1,\dots,x^n) = \left(\frac{2r^2x^2}{r^2+|\boldsymbol{x}|^2},\dots,\frac{2r^2x^n}{r^2+|\boldsymbol{x}|^2}, r\frac{|\boldsymbol{x}|^2-r^2}{|\boldsymbol{x}|^2+r^2}\right),$$

on the domain of this function, which is the set of $(x^1,\dots,x^n)$ for which $x^1>0$. Again, this function is obviously analytic.

PROBLEM 4.2. Verify that the inverse ψ_2^{-1} for the second chart given on the torus $\mathbb{T}^2$ is the one stated in the text.

Solution: We are to prove that $\psi_2^{-1}(y^1,y^2) = \big(2-\cos(y^2), y^1, \sin(y^2)\big)$. To do so, we note that

$$\begin{aligned}\psi_2\big(2\cos(y^2), y^1, \sin(y^2)\big) &= \left(y^1, 2\arctan\Big(\frac{\sin(y^2)}{3-\big(2-\cos(y^2)\big)}\Big)\right)\\ &= \left(y^1, 2\arctan\Big(\frac{\sin(y^2)}{1+\cos(y^2)}\Big)\right).\end{aligned}$$

Thus, we need to show that

$$2\arctan\Big(\frac{\sin(y^2)}{1+\cos(y^2)}\Big) = y^2\,.$$

To that end, let the left-hand side of the required equality be denoted θ. We then have

$$\tan\Big(\frac{\theta}{2}\Big) = \frac{\sin(y^2)}{1+\cos(y^2)}\,,$$

which, when squared, yields

$$\frac{1-\cos\theta}{1+\cos\theta} = \left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)^2 = \frac{1-\cos^2(y^2)}{\big(1+\cos(y^2)\big)^2} = \frac{1-\cos(y^2)}{1+\cos(y^2)}\,.$$

This equation in turn says, when denominators are cleared and cancelations performed, that

$$\cos(y^2) - \cos\theta = \cos\theta - \cos(y^2)\,,$$

In other words, $\cos(\theta) = \cos(y^2)$, so that $\theta = \pm y^2$. But, as the original equation shows, θ and y^2 have the same sign, so that it is the positive sign that is needed here. It follows that $\theta = y^2$, as required.

Problem 4.3. Verify that the sphere $\mathbb{S}^n(r)$ is orientable. To do so, modify the mapping ψ_S to be

$$\widetilde{\psi}_S(\boldsymbol{\xi}) = \Big(\frac{-r\xi^1}{r+\xi^{n+1}}, \frac{r\xi^2}{r+\xi^{n+1}}, \dots, \frac{r\xi^n}{r+\xi^{n+1}}\Big)\,,$$

and show that the Jacobian of the mapping $\widetilde{\psi}_S \circ \psi_N$ is positive at every point.

Solution: Here $\widetilde{\psi}_S$ has the same domain as ψ_S, namely the set of $\boldsymbol{\xi}$ for which $\xi^{n+1} > -1$. Since the Jacobian determinant of $\widetilde{\psi}_S \circ \psi_N^{-1}$ is the negative of the Jacobian determinant of $\psi_S \circ \psi_N^{-1}$, we might as well just prove that the latter is negative at every point where it is defined, which is to say, at every point except the origin in $\mathbb{R}^n$. Thus, let

$$(y^1,\dots,y^n) = \boldsymbol{y} = \psi_S\circ\psi_N^{-1}(\boldsymbol{x}) = r^2\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2} = $$
$$= r^2\Big(\frac{x^1}{(x^1)^2+\cdots+(x^n)^2}, \dots, \frac{x^n}{(x^1)^2+\cdots+(x^n)^2}\Big)\,.$$

By direct computation, the Jacobian determinant $\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \frac{\partial(y^1,\dots,y^n)}{\partial(x^1,\dots,x^n)}$ is the determinant of the matrix

$$\begin{pmatrix}
r^2\frac{-2(x^1)^2+|\boldsymbol{x}|^2}{|\boldsymbol{x}|^4} & r^2\frac{-2x^1x^2}{|\boldsymbol{x}|^4} & r^2\frac{-2x^2x^3}{|\boldsymbol{x}|^4} & \cdots & r^2\frac{-2x^1x^n}{|\boldsymbol{x}|^4}\\
r^2\frac{-2x^2x^1}{|\boldsymbol{x}|^4} & r^2\frac{-2(x^2)^2+|\boldsymbol{x}|^2}{|\boldsymbol{x}|^4} & r^2\frac{-2x^2x^3}{|\boldsymbol{x}|^4} & \cdots & r^2\frac{-2x^2x^n}{|\boldsymbol{x}|^4}\\
r^2\frac{-2x^3x^1}{|\boldsymbol{x}|^4} & r^2\frac{-2x^3x^2}{|\boldsymbol{x}|^4} & r^2\frac{-2(x^3)^2+|\boldsymbol{x}|^2}{|\boldsymbol{x}|^4} & \cdots & r^2\frac{-2x^3x^n}{|\boldsymbol{x}|^4}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
r^2\frac{-2x^nx^1}{|\boldsymbol{x}|^4} & r^2\frac{-2x^nx^2}{|\boldsymbol{x}|^4} & r^2\frac{-2x^nx^3}{|\boldsymbol{x}|^4} & \cdots & r^2\frac{-2(x^n)^2+|\boldsymbol{x}|^2}{|\boldsymbol{x}|^4}
\end{pmatrix}.$$

By factoring out $2r^2/|\boldsymbol{x}|^4$ from every row, we see that this determinant is

$$\Big(\frac{2r^2}{|\boldsymbol{x}|^4}\Big)^n \det \begin{pmatrix} -(x^1)^2+\frac{1}{2}|\boldsymbol{x}|^2 & -x^1x^2 & \cdots & -x^1x^n \\ -x^2x^1 & -(x^2)^2+\frac{1}{2}|\boldsymbol{x}|^2 & \cdots & -x^2x^n \\ -x^3x^1 & -x^3x^2 & \cdots & -x^3x^n \\ \vdots & \vdots & \vdots & \vdots \\ -x^nx^1 & -x^nx^2 & \cdots & -(x^n)^2+\frac{1}{2}|\boldsymbol{x}|^2 \end{pmatrix} =$$

$$= \Big(\frac{2r^2}{|\boldsymbol{x}|^4}\Big)^n \det\Big(\frac{1}{2}|\boldsymbol{x}|^2 I - M\Big),$$

where I is the $n\times n$ identity matrix and M is the symmetric matrix given by

$$\begin{pmatrix} (x^1)^2 & x^1x^2 & x^1x^3 & \cdots & x^1x^n \\ x^2x^1 & (x^2)^2 & x^2x^3 & \cdots & x^2x^n \\ x^3x^1 & x^3x^2 & (x^3)^2 & \cdots & x^3x^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x^nx^1 & x^nx^2 & x^nx^3 & \cdots & (x^n)^2 \end{pmatrix}$$

Thus, the determinant in question is the positive quantity $(2r^2/|\boldsymbol{x}|^4)^n$ times the value of the characteristic polynomial of M at the value $\frac{1}{2}|\boldsymbol{x}|^2$. Now, if the matrix M is regarded as an operator on $\mathbb{R}^n$ in the standard basis, its action is simply described as

$$M(\boldsymbol{y}) = (\boldsymbol{x}\cdot\boldsymbol{y})\boldsymbol{x}\,,$$

that is,

$$M\begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ \vdots \\ y^n \end{pmatrix} = (x^1y^1+\cdots+x^ny^n)\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ \vdots \\ x^n \end{pmatrix}.$$

Thus for any vector $\boldsymbol{y}$ in the $(n-1)$-dimensional subspace of vectors orthogonal to $\boldsymbol{x}$ (that is, $(\boldsymbol{y}\cdot\boldsymbol{x})=0$) we have $M(\boldsymbol{y})=0$, so that 0 is an eigenvalue of multiplicity $(n-1)$ for the operator represented by M. In the complementary one-dimensional subspace of vectors parallel to $\boldsymbol{x}$ (the set of scalar multiples $t\boldsymbol{x}$) we have

$$M(t\boldsymbol{x}) = (t\boldsymbol{x}\cdot\boldsymbol{x})\boldsymbol{x} = |\boldsymbol{x}|^2(t\boldsymbol{x})\,,$$

and so $|\boldsymbol{x}|^2$ is the remaining eigenvalue, of multiplicity 1.

It follows that the characteristic polynomial of M is $f(\lambda)=\lambda^{n-1}(\lambda-|\boldsymbol{x}|^2)$. The value of this characteristic polynomial at $\lambda=\frac{1}{2}|\boldsymbol{x}|^2$ is

$$-\Big(\frac{|\boldsymbol{x}|^2}{2}\Big)^n,$$

which is always negative.

To summarize, the Jacobian determinant of $\psi_S\circ\psi_N^{-1}$ is

$$-\Big(\frac{r^2}{|\boldsymbol{x}|^2}\Big)^n,$$

and the Jacobian determinant of $\widetilde{\psi}_S\circ\psi_N^{-1}$ is the negative of this, that is,

$$\Big(\frac{r^2}{|\boldsymbol{x}|^2}\Big)^n,$$

which is always positive. Since the domains of the two charts cover $\mathbb{S}^n(r)$, it follows that $\mathbb{S}^n(r)$ is orientable.

PROBLEM 4.4. In connection with the topology on the real projective plane $\mathbb{P}^2$, prove that $\tilde{U} \cap \tilde{V} = \tilde{W}$, where $W = (U \cap V) \cup \big(U \cap (-V)\big) \cup \big((-U) \cap V\big) \cup \big((-U) \cap (-V)\big)$. Since W is an open set in $\mathbb{S}^2(1)$ if U and V are, it follows that the intersection of two open sets in $\mathbb{P}^2$ is open. The facts that $\varnothing$, $\mathbb{P}^2$, and the union of any collection of open sets are all open are trivial.

Solution: Suppose $\{\boldsymbol{\xi}, -\boldsymbol{\xi}\}$ belongs to $\tilde{U}$ and $\tilde{V}$. That means that $\boldsymbol{\xi}$ belongs to either U or $-U$ and also to either V or $-V$. And hence, it belongs to W, so that $\{\boldsymbol{\xi}, -\boldsymbol{\xi}\} \in \tilde{W}$. Conversely, if $\boldsymbol{\xi} \in W$, then $\boldsymbol{\xi}$ belongs to one of the four intersections that make up the set W. Without loss of generality, assume that $\boldsymbol{\xi} \in U \cap (-V)$. That is, $\boldsymbol{\xi} \in U$, so that $\{\boldsymbol{\xi}, -\boldsymbol{\xi}\} \in \tilde{U}$, and $-\boldsymbol{\xi} \in V$, so that again $\{\boldsymbol{\xi}, -\boldsymbol{\xi}\} = \{-\boldsymbol{\xi}, \boldsymbol{\xi}\} \in \tilde{V}$, and therefore $\{\boldsymbol{\xi}, -\boldsymbol{\xi}\} \in \tilde{U} \cap \tilde{V}$. The other three cases are handled similarly. The closure of the topology under arbitrary unions is an immediate consequence of the simple fact that

$$\bigcup_{a \in A} U_a \cup \left(-\bigcup_{a \in A} U_a\right) = \bigcup_{a \in A} \big(U_a \cup (-U_a)\big)\,.$$

Thus, since $\{\boldsymbol{\xi}, -\boldsymbol{\xi}\} \in \widetilde{U} \Leftrightarrow \boldsymbol{\xi} \in U \cup (-U)$, we see that

$$\bigcup_{a \in A} \tilde{U}_a = \widetilde{\bigcup_{a \in A} U_a}\,.$$

It follows that a union of open sets is open.

PROBLEM 4.5. Consider the charts ψ_1, and ψ_3 that form part of the differentiable structure on the real projective plane. Since

$$\psi_1 \circ \psi_3^{-1}\big(\psi_3 \circ \psi_1^{-1}(\boldsymbol{y})\big) = \boldsymbol{y}$$

on the domain of this function, which is the open unit disk with the axes removed, that is, the set of $\boldsymbol{y} = (y^1, y^2)$ such that $(y^1)^2 + (y^2)^2 < 1$ and $y^1 \neq 0 \neq y^2$, we must have

$$J(\psi_1 \circ \psi_3^{-1})\big(\psi_3 \circ \psi_1^{-1}(\boldsymbol{y})\big) \cdot J(\psi_3 \circ \psi_1^{-1})(\boldsymbol{y}) = 1\,.$$

Here $J(\psi)$ denotes the Jacobian determinant of any mapping $\psi : \mathbb{R}^n \to \mathbb{R}^n$. Verify this equation.

Solution: A not-too-difficult computation reveals that this product is

$$\frac{\operatorname{sgn}(y^2)\sqrt{1-(y^1)^2-(y^2)^2}}{\sqrt{1-\big(1-(y^1)^2-(y^2)^2\big)-(y^1)^2}} \cdot \frac{y^2}{\sqrt{1-(y^1)^2-(y^2)^2}}\,.$$

It is then easy to see that this is

$$\frac{\operatorname{sgn}(y^2)y^2}{\sqrt{(y^2)^2}} = \frac{|y^2|}{|y^2|} = 1\,.$$

PROBLEM 4.6. Consider the *real projective line* $\mathbb{P}^1$, a point of which is a two-element set whose elements are antipodal points on the unit circle. That is, a point is $\{\boldsymbol{\xi}, -\boldsymbol{\xi}\} = \big\{(\xi^1, \xi^2), (-\xi^1, -\xi^2)\big\}$, where $(\xi^1)^2 + (\xi^2)^2 = 1$. Given the topology analogous to the one on $\mathbb{P}^2$, we define two open sets that cover $\mathbb{P}^1$, namely $U_1 =$

$\big\{\{\boldsymbol{\xi},-\boldsymbol{\xi}\} : \xi^1 \neq 0\big\}$ and $U_2 = \big\{\{\boldsymbol{\xi},-\boldsymbol{\xi}\} : \xi^2 \neq 0\big\}$ and charts $\psi_1 : U_1 \to (-1,1)$ and $\psi_2 : U_2 \to (-1,1)$ by

$$\begin{aligned}\psi_1\big(\{\boldsymbol{\xi},-\boldsymbol{\xi}\}\big) &= \operatorname{sgn}(\xi^1)\xi^2\\ \psi_2\big(\{\boldsymbol{\xi},-\boldsymbol{\xi}\}\big) &= \operatorname{sgn}(\xi^2)\xi^1 .\end{aligned}$$

Describe $\psi_1^{-1}(y)$, $\psi_2^{-1}(y)$, $\psi_2\circ\psi_1^{-1}(y)$, and $\psi_1\circ\psi_2^{-1}(y)$ explicitly. How smooth is $\mathbb{P}^1$? Is it orientable?

Also, show that the mapping $f\big(\{\boldsymbol{\xi},-\boldsymbol{\xi}\}\big) = \big((\xi^1)^2-(\xi^2)^2, 2\xi^1\xi^2\big) = \boldsymbol{\eta}$ is a one-to-one mapping of $\mathbb{P}^1$ onto the unit circle and has inverse

$$f^{-1}(\boldsymbol{\eta}) = f^{-1}(\eta^1,\eta^2) = \\ = \left\{\Big(\sqrt{\frac{1+\eta^1}{2}}, \operatorname{sgn}(\eta^2)\sqrt{\frac{1-\eta^1}{2}}\Big), \Big(-\sqrt{\frac{1+\eta^1}{2}}, -\operatorname{sgn}(\eta^2)\sqrt{\frac{1-\eta^1}{2}}\Big)\right\}.$$

How smooth are the mappings f and f^{-1}?

Solution: It is easy to show that

$$\begin{aligned}\psi_1^{-1}(y) &= \big\{\big(\sqrt{1-y^2},y\big),\big(-\sqrt{1-y^2},-y\big)\big\}\\ \psi_2^{-1}(y) &= \big\{y,\sqrt{1-y^2}\big),\big(-y,-\sqrt{1-y^2}\big)\big\}.\end{aligned}$$

It then follows from the definition that $\psi_2\circ\psi_1^{-1}(y) = \operatorname{sgn}(y)\sqrt{1-y^2} = \psi_1\circ\psi_2^{-1}(y)$ on its domain, characterize by the conditions $0 < |y| < 1$. This mapping has Jacobian determinant $-|y|/\sqrt{1-y^2}$, which is negative. However, if we modify ψ_1 by writing $\widetilde{\psi}_1\big(\{\boldsymbol{\xi},-\boldsymbol{\xi}\}\big) = -\operatorname{sgn}(\xi^1)\xi^2$, then the pair of mappings $\widetilde{\psi}_1$ and ψ_2 provide transition mappings whose common Jacobian determinant is positive. Hence $\mathbb{P}^1$ is orientable. (In general, if a manifold is covered by the domains of two charts and the transition Jacobian determinant is of constant sign on the intersection of the two domains, then the manifold is orientable, since it is always possible to modify one of the charts by changing the sign of one of its components and thereby ensure that the Jacobian is always positive.)

Given that $1 = |\boldsymbol{\xi}|^2 = (\xi^1)^2+(\xi^2)^2$, it immediately follows that $|\boldsymbol{\eta}|^2 = \big|f\big(\{\boldsymbol{\xi},-\boldsymbol{\xi}\}\big)|^2 = \big((\xi^1)^2-(\xi^2)^2\big)^2 + 4(\xi^1)^2(\xi^2)^2 = \big((\xi^1)^2+(\xi^2)^2\big)^2 = 1^2 = 1$, so that indeed $\boldsymbol{\eta}$ belongs to $\mathbb{S}^1(1)$. Given that $\boldsymbol{\xi} = \pm\Big(\sqrt{\frac{1+\eta^1}{2}}, \operatorname{sgn}(\eta^2)\sqrt{\frac{1-\eta^1}{2}}\Big)$ it follows that $(\xi^1)^2-(\xi^2)^2 = \eta^1$ and $2\xi^1\xi^2 = \operatorname{sgn}(\eta^2)\sqrt{1-(\eta^1)^2} = \operatorname{sgn}(\eta^2)|\eta^2| = \eta^2$. Thus, the two mappings really are inverses of each other.

Moreover,

$$\begin{aligned}\psi_N\Big(f\big(\psi_1^{-1}(y)\big)\Big) &= \psi_N\Big(f\big(\{\sqrt{1-y^2},y),(-\sqrt{1-y^2},-y)\}\big)\Big)\\ &= \psi_N\big(1-2y^2, 2y\sqrt{1-y^2}\big)\\ &= \frac{1-2y^2}{2y\sqrt{1-y^2}},\end{aligned}$$

and this function is analytic on its domain, which is $(-1,0)\cup(0,1)$. Replacing ψ_1 by ψ_2 or ψ_N by ψ_S would lead to a similar result. Hence $\mathbb{P}^1$ is an analytic manifold and is in fact diffeomorphic to the unit circle $\mathbb{S}^1(1)$.

PROBLEM 4.7. In Example 4.7, transport the vector $(0,1)$ along the leg P_1P_2 of the geodesic triangle using r as the parameter instead of arc length s, as was done in the text. (This is much simpler than using s.)

Solution: If we take $\boldsymbol{\gamma}(r)=(r,0)$, then $\boldsymbol{\gamma}'(r)=(1,0)$ and so the covariant derivative we are setting equal to zero is

$$\begin{aligned}\nabla_{(1,0)}\Big(a(r)\frac{\partial}{\partial r}+b(r)\frac{\partial}{\partial\theta}\Big) &= \big(a'(r)+a(r)\Gamma^1_{11}+b(r)\Gamma^1_{12}\big)\frac{\partial}{\partial r}+\big(b'(r)+a(r)\Gamma^2_{11}+b(r)\Gamma^2_{12}\big)\frac{\partial}{\partial\theta}\\ &= \Big(a'(r)-\frac{a(r)}{r}\Big)\frac{\partial}{\partial r}+\Big(b'(r)+\frac{b(r)}{r}\Big)\frac{\partial}{\partial\theta}\,.\end{aligned}$$

The resulting differential equations $a'(r)-a(r)/r=0$, $b'(r)+b(r)/r=0$, together with the initial conditions $a(k/5)=0$, $b(k/5)=1$ then quickly yield the solutions $a(r)\equiv 0$, $b(r)\equiv k/5r$, so that, as in the text, the vector becomes $(0,1/2)$ at $r=2k/5$.

PROBLEM 4.8. Verify the equation

$$X(f)=\boldsymbol{X}\cdot\nabla\tilde{f}\,.$$

Solution: This is precisely the chain rule. If $\boldsymbol{X}=a\frac{\partial\boldsymbol{r}}{\partial u}+b\frac{\partial\boldsymbol{r}}{\partial v}$, then

$$\begin{aligned}X(f) &= a\frac{\partial f}{\partial u}+b\frac{\partial f}{\partial v)}\\ &= a\frac{\partial\tilde{f}(\boldsymbol{r}(u,v))}{\partial u}+b\frac{\partial\tilde{f}(\boldsymbol{r}(u,v))}{\partial v}\\ &= a\nabla\tilde{f}\cdot\frac{\partial\boldsymbol{r}}{\partial u}+b\nabla\tilde{f}\cdot\frac{\partial\boldsymbol{r}}{\partial v}\\ &= \nabla\tilde{f}\cdot\Big(a\frac{\partial\boldsymbol{r}}{\partial u}+b\frac{\partial\boldsymbol{r}}{\partial v}\Big)\,.\end{aligned}$$

PROBLEM 4.9. A vector $\boldsymbol{a}=(a^1,a^2,a^3)$ in $\mathbb{R}^3$ can be naturally associated with a skew-symmetric 3×3 matrix

$$\boldsymbol{A}=\begin{pmatrix}0 & -a^3 & a^2\\ a^3 & 0 & -a^1\\ -a^2 & a^1 & 0\end{pmatrix}.$$

Show that, if $\boldsymbol{b}=(b^1,b^2,b^3)$ is associated in this way with the matrix $\boldsymbol{B}$, then the cross product $\boldsymbol{a}\times\boldsymbol{b}$ is associated with $[\boldsymbol{A},\boldsymbol{B}]=\boldsymbol{A}\,\boldsymbol{B}-\boldsymbol{B}\,\boldsymbol{A}$. (Replacing an associative product with its commutator, that is, replacing AB with $[A,B]$ is a standard way of turning an associative algebra into a Lie algebra. If the algebra happens to be commutative, of course, the Lie algebra is trivial, since all products are equal to zero.)

Solution: This is straightforward computation.

PROBLEM 4.10. Verify that the triangle $P_1P_2P_3$ on the pseudo-hemisphere is a right triangle by showing that

$$\cosh\left(\frac{1}{2}\ln\Big(\frac{553+25\sqrt{481}}{72}\Big)\right)=\cosh\big(\ln(2)\big)\,\cosh\big(\ln(3)\big)=\frac{25}{12}\,.$$

Solution: The simplest thing is to let *Mathematica* do the work. The command `FullSimplify[Cosh[Log[Sqrt[(553 + 25 Sqrt[481])/72]]]]` produces the output 25/12. It is quite trivial to compute that the other equation is also satisfied.

APPENDIX 5

Differential Equations

PROBLEM 5.1. Prove the basic facts about integrals of vector-valued functions, that is, that the norm of the integral is not greater than the integral of the norm and that the derivative of an indefinite integral is the integrand.

Solution: For the derivatives, this result is a simple matter of definition: Differentiation (and integration) of a vector-valued function is carried out by performing the corresponding operation on each component, and continuity of such a function means simply that each component is continuous. Therefore, integrating a (continuous) vector-valued function from a to x and then differentiating with respect to x simply leads back to the integrand. The converse form of the fundamental theorem of calculus also holds: If a function $\boldsymbol{f}(t)$ has a continuous derivative $\boldsymbol{f}'(t)$, then the integral of $\boldsymbol{f}'(t)$ from a to b is $\boldsymbol{f}(b) - \boldsymbol{f}(a)$. Again, this is just a matter of applying the known theorems to each component.

The norm inequality for the integral is only slightly more complicated. Let $\boldsymbol{f}(\boldsymbol{x}) = \big(f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})\big)$. Then, since we are going to deal only with continuous functions, we can assume that our integrals are Riemann integrals, approximated by Riemann sums:

$$\int\limits_R \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} \approx \sum c_k \boldsymbol{f}(\boldsymbol{x}_k)\,,$$

where the coefficients c_k are the n-dimensional volumes of the pieces into which R is partitioned to form the Riemann sum. The crucial point is that the c_k are non-negative numbers.

We thus find that

$$\left\| \sum c_k \boldsymbol{f}(\boldsymbol{x}_k) \right\| \le \sum c_k \big\| \boldsymbol{f}(\boldsymbol{x}_k) \big\|\,.$$

But this last expression is simply a Riemann sum for the integral of $\|\boldsymbol{f}(\boldsymbol{x})\|$. Hence, for all sufficiently fine partitions, it approximates the integral, and so

$$\left\| \int\limits_R \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} \right\| \le \int\limits_R \big\|\boldsymbol{f}(\boldsymbol{x})\big\|\, d\boldsymbol{x}\,.$$

PROBLEM 5.2. Consider the case $p = 1$ (real-valued functions of a real variable) and the differential equation $y'(x) = y(x)$ with initial condition $y(0) = 1$. Execute the method of successive approximations on this initial-value problem and express the solution of it as convergent infinite series. What function does this series represent?

Solution: The recursive relation is

$$\begin{aligned} y_0(x) &\equiv 1\,, \\ y_{n+1}(x) &= 1 + \int_0^x y_n(t)\,dt\,, \end{aligned}$$

so that

$$\begin{aligned} y_1(x) &= 1 + x\,, \\ y_2(x) &= 1 + x + \frac{x^2}{2}\,, \\ y_3(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\,, \end{aligned}$$

and so by a very easy induction we see that the limit of the $y_n(x)$ is

$$y(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\,,$$

which, as is well known, is the Maclaurin series of the exponential function $y(x) = e^x$.

PROBLEM 5.3. Consider an nth-order initial value problem for a real-valued function $y(x)$ consisting of a differential equation

$$y^{(n)}(x) = f\big(x, y(x), y'(x), y''(x), \ldots, y^{(n-1)}(x)\big)$$

with initial conditions $y(x_0) = y^0$, $y'(x_0) = y^1, \ldots,$ $y^{(n-1)}(x_0) = y^{n-1}$, where f is a real-valued function having continuous partial derivatives on a connected open subset U of $\mathbb{R}^{n+1}$ containing the point $(x_0, y^0, y^1, \ldots, y^{n-1})$. Prove that this problem has a unique solution if the partial derivatives of f on all of its variables except the first are uniformly bounded in some cube C in $\mathbb{R}^{n+1}$ containing the point $(x_0, y^0, y^1, \ldots, y^{n-1})$ and contained in U. *Hint:* Let $\boldsymbol{y}(x) = \big(y(x), y'(x), \ldots, y^{(n-1)}(x)\big)$.

Solution: The hint gives away the secret here, and we see that an nth-order ordinary differential equation of this form has a unique solution with prescribed initial values for the function and its first $n-1$ derivatives. All we have to do is invoke the basic Theorem 5.1.

PROBLEM 5.4. The space of continuous functions on an interval $[a, b]$ of the real line with values in $\mathbb{R}^p$ is a complete metric space under the norm

$$|\boldsymbol{y}|_\infty = \sup_{a \le x \le b} |\boldsymbol{y}(x)|\,.$$

(The proof of this fact is the simple observation that convergence of a sequence $\{\boldsymbol{y}_n\}_{n=0}^\infty$ in this norm means uniform convergence of each component y^{nj}, $1 \le j \le p$.)

Prove that if X is any complete metric space and $\varphi : X \to X$ is a *contraction*, that is, there exists a positive number $\alpha < 1$ such that for all $x \in X$ and $y \in X$, $d\big(\varphi(x), \varphi(y)\big) \le \alpha d(x, y)$, then the mapping φ has a unique fixed point z such that $\varphi(z) = z$, and it can be reached starting from any point x_0 by setting $x_{n+1} = \varphi(x_n)$, $n = 0, 1, \ldots$ and taking the limit as $n \to \infty$. Show how to derive our existence and uniqueness theorem from this fixed-point theorem by constructing a suitable X and φ.

Solution: The abstract theorem essentially amounts to a repetition of the argument in the text. Since $d\big(\varphi(x_{n+1}),\varphi(x_n)\big) \le \alpha d\big(\varphi(x_n),\varphi(x_{n-1})\big)$, it follows that

$$d\big(\varphi(x_n),\varphi(x_1)\big) \le \alpha^n d\big(\varphi(x_1),\varphi(x_0)\big)\,,$$

and hence

$$d\big(\varphi(x_m),\varphi(x_n)\big) \le d\big(\varphi(x_m),\varphi(x_1)\big)+d\big(\varphi(x_1),\varphi(x_n)\big) \le (\alpha^m+\alpha^n)d\big(\varphi(x_1),\varphi(x_0)\big)\,,$$

and this last expression is arbitrarily small for large m and n.

It follows that the sequence $\{\varphi(x_n)\}_{n=0}^{\infty}$ is a Cauchy sequence and hence has a limit z, which is therefore a fixed point, since $\varphi(z) = \varphi(\lim x_n) = \lim \varphi(x_n) = \lim(x_{n+1}) = z$.

For the specific application, we take X to be the space of continuous mappings $\boldsymbol{y} : [x_0-\varepsilon, x_0+\varepsilon] \to C$ with the given norm and φ the mapping given by

$$\big(\varphi(\boldsymbol{y})\big)(x) = \boldsymbol{y}_0 + \int_{x_0}^{x} \boldsymbol{f}\big(x,\boldsymbol{y}(x)\big)\,dx\,.$$

The argument in the text shows that this is a contraction mapping ($\alpha = 1/2$) and hence has a fixed point.

Problem 5.5. Show that the solution of the initial-value problem

$$\boldsymbol{y}(x) = \boldsymbol{y}_0 + \int_{x_0}^{x} \boldsymbol{f}\big(t,\boldsymbol{y}(t)\big)\,dt$$

actually has continuous derivatives of order $k+1$ provided all the partial derivatives of $\boldsymbol{f}(x,\boldsymbol{y})$ of order k or less are continuous.

Solution: It is easy to show by induction that $\boldsymbol{y}^{(k)}(x)$ is a sum of products of partial derivatives of $\boldsymbol{f}\big(x,\boldsymbol{y}(x)\big)$ and derivatives of $\boldsymbol{y}$ of orders at most $k-1$. Thus any smoothness in our data $\boldsymbol{f}(x,\boldsymbol{y})$ leads to "bonus" smoothness in the solution.

Problem 5.6. Prove that the system of differential equations

$$\frac{\left(\frac{\partial f}{\partial x^{n+1}}\right)^* dx^1}{M_1^*} = \cdots = \frac{\left(\frac{\partial f}{\partial x^{n+1}}\right)^* dx^n}{M_n^*}$$

satisfies the vanishing-divergence condition.

Solution: If we take account of the relations (8) and (9), we see that we need to prove that the equation

$$\sum_{j=1}^{n}\left(\frac{\partial M_{n+1}}{\partial x^j} - M_j\frac{\frac{\partial^2 f}{\partial x^j\partial x^{n+1}}}{\frac{\partial f}{\partial x^{n+1}}} - \frac{\frac{\partial f}{\partial x^j}}{\frac{\partial f}{\partial x^{n+1}}}\frac{\partial M_j}{\partial x^{n+1}} + M_j\frac{\frac{\partial f}{\partial x^j}\,\frac{\partial^2 f}{\partial (x^{n+1})^2}}{\left(\frac{\partial f}{\partial x^{n+1}}\right)^2}\right) = 0$$

holds when $x^{n+1} = h(x^1,\ldots,x^n)$. The sum on j thus consists of four terms, which we examine one at a time. By the original divergence condition, the sum over j in the first term yields simply

$$-\frac{\partial M_{n+1}}{\partial x^{n+1}}\,,$$

and this term can be incorporated into the third term by the simple device of extending j to $n+1$.

Now the fourth term can be written as

$$\frac{\frac{\partial^2 f}{\partial (x^{n+1})^2}}{\frac{\partial f}{\partial x^{n+1}}} \sum_{j=1}^{n} M_j \left(\frac{\frac{\partial f}{\partial x^j}}{\frac{\partial f}{\partial x^{n+1}}} \right),$$

which by relation (9) is

$$-\frac{\frac{\partial^2 f}{\partial (x^{n+1})^2}}{\frac{\partial f}{\partial x^{n+1}}} M_{n+1}$$

on the set where the differential equations are satisfied. Thus, this term can be adjoined to the sum in the second term by extending the index j to $n+1$. The expression can now be rearranged as

$$-\frac{1}{\left(\frac{\partial f}{\partial x^{n+1}}\right)} \sum_{j=1}^{n+1} M_j \frac{\partial^2 f}{\partial x^j \, \partial x^{n+1}} + \frac{\partial M_j}{\partial x^{n+1}} \frac{\partial f}{\partial x^j}\,.$$

This expression is equal to

$$-\frac{1}{\left(\frac{\partial f}{\partial x^{n+1}}\right)} \frac{\partial}{\partial x^{n+1}} \left(\sum_{j=1}^{n+1} M_j \frac{\partial f}{\partial x^j} \right) = 0\,,$$

which was what we needed to prove.

Problem 5.7. Consider a particle of mass M fixed in space and another of mass m going around it in a circular orbit of radius r in accordance with Newtonian physics. If the particle is at position $\boldsymbol{u}(t)$ at time t in a system of coordinates fixed in absolute space with origin at the particle of mass M, then the force on the orbiting particle is a purely centripetal force

$$\boldsymbol{F}(t) = -\frac{GMm}{|\boldsymbol{u}(t)|^3}\boldsymbol{u}(t) = -\frac{GMm}{r^3}\boldsymbol{u}(t)\,.$$

Now imagine a second set of coordinates, again with origin at the particle of mass M, but rotating in such a way that the position of the particle of mass m is constantly $\boldsymbol{v}$. Show that there will be no Coriolis force acting on the orbiting particle in this system, but there will be a centrifugal force equal and opposite to the centripetal force in the other system, and therefore the net force on the now-stationary particle is zero. (Use Kepler's third law, which states that the period of revolution is

$$T = \frac{2\pi |\boldsymbol{v}|^{3/2}}{\sqrt{GM}} = \frac{2\pi r^{3/2}}{\sqrt{GM}}\,,$$

to get the angular velocity vector $\boldsymbol{\omega}$.)

Solution: It is obvious that the Coriolis force on the particle is zero in the second frame of reference, since the velocity of the particle is zero.

It is also obvious that the second frame of reference is rotating with the same speed and about the same axis as the particle itself. Without any loss of generality then, we can assume that the position of the particle at time t in the absolute frame of reference is $\boldsymbol{u}(t) = r\cos\omega t\,\boldsymbol{i} + r\sin\omega t\,\boldsymbol{j}$, where

$$\omega = \frac{2\pi}{T} = \frac{\sqrt{GM}}{r^{3/2}}$$

Thus the angular velocity of the second frame relative to the first is just $\boldsymbol{\omega} = \omega\,\boldsymbol{k}$. Without any loss of generality, we can assume that the two frames of reference

coincide at time $t = 0$, so that $\boldsymbol{v} = r\boldsymbol{i}$. The centrifugal force on the particle in the rotating frame of reference is then

$$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{v}) = -mr\boldsymbol{\omega} \times \boldsymbol{j} = mr\omega^2 \boldsymbol{i}\,.$$

When we substitute the value just obtained for ω, we see that this force is

$$\frac{GMm}{r^3}\boldsymbol{v}\,,$$

and so it exactly cancels the centripetal force due to gravity, leaving the particle fixed.

Problem 5.8. Suppose a particle moves westward along the earth's equator. What linear speed will cause the Coriolis force to cancel the centrifugal force? (The earth's angular velocity of rotation is 7.29×10^{-5} radians per second. Take the altitude of the particle (radius of the earth) to be 6.37×10^6 meters.)

Solution: If we let the position of the particle be $\boldsymbol{v} = r\,\boldsymbol{i}$ and its velocity $\boldsymbol{v}' = -v\,\boldsymbol{j}$, while $\boldsymbol{\omega} = \omega\,\boldsymbol{k}$, setting the centrifugal force equal to the negative of the Coriolis force gives the equation

$$-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{v}) = 2\boldsymbol{\omega} \times \boldsymbol{v}'\,,$$

which translates to $2v\omega = r\omega^2$, or $v = r\omega/2$, which is about 232 meters per second.

Problem 5.9. Carry out the computation that shows the conservation of energy and the vertical component of angular momentum in the general case of a rotating rigid body subject only to gravitational forces.

Solution: These are very routine computations, and we shall give just the one for the total energy

$$E = \frac{1}{2}\big(Ap^2 + Bq^2 + Cr^2\big) + Mg(\upsilon_{c1}r_{31} + \upsilon_{c2}r_{32} + \upsilon_{c3}r_{33})\,.$$

(The coefficient of Mg here is the z-coordinate of the center of gravity of the body in coordinates fixed in space. Thus, this term represents potential energy, while the first term represents the kinetic energy.)

We find by routine calculus that

$$\begin{aligned}
&dE = Ap\,dp + Bq\,dq + Cr\,dr + Mg(\upsilon_{c1}\,dr_{31} + \upsilon_{c2}\,dr_{32} + \upsilon_{c3}\,dr_{33}) = \\
&= p\big((B-C)qr - Mg(\upsilon_{c2}r_{33} - \upsilon_{c3}r_{32}\big) + q\big((C-A)rp - Mg(\upsilon_{c3}r_{31} - \upsilon_{c1}r_{33}\big) + \\
&\qquad + r\big((A-B)pq - Mg(\upsilon_{c1}r_{32} - \upsilon_{c2}r_{31}\big) + \\
&\qquad + Mg\big(\upsilon_{c1}(r_{32}r - r_{33}q) + \upsilon_{c2}(r_{33}p - r_{31}r) + \upsilon_{c3}(r_{31}q - r_{32}p)\big)\,.
\end{aligned}$$

The problem is now just a matter of canceling terms in pairs, which is exceedingly easy to do.

The vertical component of the angular momentum is given by $\Omega = Apr_{31} + Bqr_{32} + Crr_{33}$, and the computation is equally simple in this case.

Problem 5.10. Use the law of conservation of energy as the integral and follow the last-multiplier principle to solve the first three equations in the Euler case problem in closed form.

Solution: We can write the first three equations in the form

$$\begin{aligned}
\frac{dp}{dt} &= \frac{(B-C)qr}{A}\,,\\
\frac{dq}{dt} &= \frac{(C-A)rp}{B}\,,\\
\frac{dr}{dt} &= \frac{(A-B)pq}{C}\,.
\end{aligned}$$

or, once again eliminating dt,

$$\frac{A\,dp}{(B-C)qr} = \frac{B\,dq}{(C-A)rp} = \frac{C\,dr}{(A-B)pq}\,.$$

All we need to do is scrounge up one integral, and Jacobi's method will provide a second integral and thereby solve the system. This is ludicrously easy to do in this case, and also unnecessary, since one can easily separate the variables in any of these three equations. Let us see, however, what the last-multiplier principle says in this case. Our integral $f(p,q,r)$ is $Ap^2 + Bq^2 + Cr^2 = 2E$, which can be solved for r. It turns out to be unnecessary to do so, however, since the integrating factor is $\partial f/\partial r = 2Cr$. Since a constant factor has no effect on an integrating factor, we can just use r as the integrating factor for the first equation in the pair.

Thus, the last-multiplier principle says that the equation

$$\frac{A\,dp}{(B-C)q} = \frac{B\,dq}{(C-A)p}$$

is exact. But we could have seen that immediately, since we could have simply canceled r in the first equation and then separated the variables to get

$$A(C-A)p^2 = B(B-C)q^2 + K\,,$$

for some constant K. This says $A^2p^2 + B^2q^2 + K = C(Ap^2 + Bq^2) = C(2E - Cr^2)$, which in turn can be written as $A^2p^2 + B^2q^2 + C^2r^2 = 2EC - K = L$. Thus, the solution in this case is the trajectory that is the intersection of two ellipsoids

$$\begin{aligned}
Ap^2 + Bq^2 + Cr^2 &= 2E\,,\\
A^2p^2 + B^2q^2 + C^2r^2 &= L\,.
\end{aligned}$$

PROBLEM 5.11. Combining the Euler and Lagrange cases by assuming that $\boldsymbol{v}_c = \mathbf{0}$ and $A = B$, show that the solution in this case can be expressed in terms of elementary functions. (The rotation of the earth falls under this case, to a fair degree of approximation.)

Solution: On the one hand, the third Euler equation in this case implies that r is constant. Then conservation of energy implies that $p^2 + q^2$ is also constant. Thus, the trajectory of the solution is the intersection of a cylinder with a plane perpendicular to it. In particular, the angular speed is constant, and the axis of rotation precesses at a constant rate. In the case of the earth, the axis of rotation takes about 25,000 years to describe a circle in the sky. The equinoxes precess by about one degree in an average human lifetime.

PROBLEM 5.12. To solve a second-order linear homogeneous[1] differential equation $y''(x) + F(x)y'(x) + G(x)y(x) = 0$, let $z(x) = y'(x)$ and write the equation as a system of two equations:

$$\begin{aligned} y'(x) &= z(x)\,, \\ z'(x) &= -G(x)y(x) - F(x)G(x)\,. \end{aligned}$$

Use the method of successive approximations to produce the solution of the initial-value problem $y''(x) + y(x) = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution: We have $y'(x) = z(x)$ and $z'(x) = -y(x)$. As a result, we have the matrix recursion equation

$$\begin{pmatrix} y_{n+1}(x) \\ z_{n+1}(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^x \begin{pmatrix} z_n(x) \\ -y_n(x) \end{pmatrix} dx\,.$$

By simple computation, we get successively

$$\begin{aligned} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} &= \begin{pmatrix} 1 \\ -x \end{pmatrix}, \\ \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} &= \begin{pmatrix} 1 - \frac{1}{2}x^2 \\ -x \end{pmatrix}, \\ \begin{pmatrix} y_3 \\ z_3 \end{pmatrix} &= \begin{pmatrix} 1 - \frac{1}{2}x^2 \\ -x + \frac{1}{3!}x^3 \end{pmatrix}, \\ \begin{pmatrix} y_4 \\ z_4 \end{pmatrix} &= \begin{pmatrix} 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 \\ -x + \frac{1}{3!}x^3 \end{pmatrix}. \end{aligned}$$

We recognize here the Maclaurin series for $\cos x$ and $-\sin x$, which are of course the exact solution of this system.

[1]The word *homogeneous* here should not be confused with the use of that word in classifying first-order differential equations, as was done above. All it means is that the "forcing function" that would ordinarily appear on the right-hand side is zero.

APPENDIX 6

Invariance

Problem 6.1. Verify that the two parametrizations of the hemisphere given in this chapter have the transition mappings indicated. That is, show that, for example, the second equation becomes the identity $v = v$, when the values of x and y given in the third and fourth equations are substituted into the second equation.

Solution: For easy reference, we restate the transition equations here:

$$\begin{aligned} u &= \arccos\left(\frac{\sqrt{x^2+y^2}}{R}\right), \\ v &= 2\arctan\left(\frac{y}{x+\sqrt{x^2+y^2}}\right), \\ x &= R\cos u\, \cos v\,, \\ y &= R\cos u\, \sin v\,. \end{aligned}$$

Doing just the example of the second equation (the others are handled the same way), we have

$$\begin{aligned} v &= 2\arctan\left(\frac{y}{x+\sqrt{x^2+y^2}}\right) \\ \tan\left(\frac{v}{2}\right) &= \frac{R\cos u\, \sin v}{R\cos u\, \cos v + R\cos u}\Bigg) \\ &= \frac{\sin v}{\cos v + 1}\,. \end{aligned}$$

Here, we have made use of the assumption $0 < u < \pi/2$, so that $\cos u > 0$. That justified removing the square root in the second equation and canceling $R\cos u$ in the third. What remains here is easily seen to be a trigonometric identity. First of all both sides of the equation are positive for $v > 0$ and negative for $v < 0$. Hence it suffices to prove that their squares are equal. At that point, one has only to replace $\tan^2(v/2)$ by $(1-\cos v)/(1+\cos v)$ and $\sin^2 v$ by $1-\cos^2 v$ to get an obvious identity.

Problem 6.2. Verify the relations between the metric coefficients of A and those of B as given in the text.

Solution: These are quite straightforward, and not really complicated. As shown in the text, we have

$$
\begin{aligned}
E &= \frac{\partial \boldsymbol{r}}{\partial u} \cdot \frac{\partial \boldsymbol{r}}{\partial u} = \Big(\frac{\partial \boldsymbol{s}}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \boldsymbol{s}}{\partial y}\frac{\partial y}{\partial u}\Big) \cdot \Big(\frac{\partial \boldsymbol{s}}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \boldsymbol{s}}{\partial y}\frac{\partial y}{\partial u}\Big) \\
&= \frac{\partial \boldsymbol{s}}{\partial x} \cdot \frac{\partial \boldsymbol{s}}{\partial x}\Big(\frac{\partial x}{\partial u}\Big)^2 + 2\frac{\partial \boldsymbol{s}}{\partial x} \cdot \frac{\partial \boldsymbol{s}}{\partial y}\frac{\partial x}{\partial u}\frac{\partial y}{\partial u} + \frac{\partial \boldsymbol{s}}{\partial y} \cdot \frac{\partial \boldsymbol{s}}{\partial y}\Big(\frac{\partial y}{\partial u}\Big)^2 \\
&= P\Big(\frac{\partial x}{\partial u}\Big)^2 + 2Q\,\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial u} + R\Big(\frac{\partial y}{\partial u}\Big)^2 .
\end{aligned}
$$

The other relations are handled similarly.

Problem 6.3. Verify the relation between the two first fundamental forms of A and B as given in the text.

Solution: The computations for this result get messy, although they retain sufficient symmetry to make it easy to navigate around in them. Everything depends on the two Jacobian matrices being inverses of each other, that is, we need the four equations

$$
\begin{aligned}
\frac{\partial x}{\partial u}\,\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\,\frac{\partial v}{\partial x} &= 1\,, \\
\frac{\partial y}{\partial u}\,\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\,\frac{\partial v}{\partial x} &= 0\,, \\
\frac{\partial x}{\partial u}\,\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\,\frac{\partial v}{\partial y} &= 0\,, \\
\frac{\partial y}{\partial u}\,\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\,\frac{\partial v}{\partial y} &= 1\,.
\end{aligned}
$$

If we combine the equations relating the two fundamental forms with the equations relating the pairs of differentials, that is,

$$
\begin{aligned}
du &= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\,, \\
dv &= \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy\,, \\
dx &= \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\,, \\
dy &= \frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv\,,
\end{aligned}
$$

we arrive at the following rather messy expression for $E\,du^2 + 2F\,du\,dv + G\,dv^2$ in terms of P, Q, R, dx, and dy:

$$\begin{aligned}
&E\,du^2 + 2F\,du\,dv + G\,dv^2 = \\
&\left[P\Bigl(\frac{\partial x}{\partial u}\Bigr)^2 + 2Q\,\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial u} + R\Bigl(\frac{\partial y}{\partial u}\Bigr)^2\right]\left[\Bigl(\frac{\partial u}{\partial x}\Bigr)^2 dx^2 + 2\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y}\,dx\,dy + \Bigl(\frac{\partial u}{\partial y}\Bigr)^2\right] + \\
&\qquad + 2\left[P\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v} + Q\Bigl(\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial v} + \frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u}\Bigr) + R\frac{\partial y}{\partial u}\,\frac{\partial y}{\partial v}\right]\times \\
&\qquad \times\left[\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x}\,dx^2 + \Bigl(\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x}\,dx\,dy + \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y}\,dv^2\right] + \\
&+\left[P\Bigl(\frac{\partial x}{\partial v}\Bigr)^2 + 2Q\,\frac{\partial x}{\partial v}\,\frac{\partial y}{\partial v} + R\Bigl(\frac{\partial y}{\partial v}\Bigr)^2\right]\left[\Bigl(\frac{\partial v}{\partial x}\Bigr)^2 dx^2 + 2\frac{\partial v}{\partial x}\,\frac{\partial v}{\partial y}\,dx\,dy + \Bigl(\frac{\partial v}{\partial y}\Bigr)^2\right].
\end{aligned}$$

If expanded fully, this expression would contain 34 terms. Every one of them is handled in the same way, however, and when we group them as differentials, we get only three terms, the coefficients of dx^2, $dx\,dy$, and dy^2. The coefficient of dx^2, for example, can be computed as

$$\begin{aligned}
&P\Bigl(\frac{\partial x}{\partial u}\,\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\,\frac{\partial v}{\partial x}\Bigr)^2 + 2Q\Bigl(\frac{\partial x}{\partial u}\,\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\,\frac{\partial v}{\partial x}\Bigr)\times \\
&\qquad\qquad \times\Bigl(\frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\,\frac{\partial v}{\partial x}\Bigr) + R\Bigl(\frac{\partial y}{\partial u}\,\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\,\frac{\partial v}{\partial x}\Bigr)^2,
\end{aligned}$$

From the Jacobian matrix relations, then the coefficient of dx^2 is

$$P \cdot 1^2 + 2Q \cdot 1 \cdot 0 + R \cdot 0^2 = P\,.$$

In exactly the same way we find that the coefficient of $dx\,dy$ is $2Q$, and the coefficient of dy^2 is R.

PROBLEM 6.4. Illustrate the results of the last two problems by considering the two parametrizations of the hemisphere of radius R given as an example. Find the length of the portion of a great circle joining the points $(0.64R, 0.60R, 0.48R)$ and $(-0.64R, 0.60R, 0.48R)$ and the area between this great circle and the equator.

Solution: Let us denote the metric coefficients of the parametrization $\boldsymbol{r}(u, v)$ as usual by E, F, and G, and those of the mapping $\boldsymbol{s}(x, y)$ by $\widetilde{E}$, $\widetilde{F}$, $\widetilde{G}$. We compute in a very straightforward manner that

$$\begin{aligned}
E &= R^2\,, \\
F &= 0\,, \\
G &= R^2\cos^2(u)\,, \\
\widetilde{E} &= \frac{R^2 - y^2}{R^2 - x^2 - y^2}\,, \\
\widetilde{F} &= \frac{xy}{R^2 - x^2 - y^2} \\
\widetilde{G} &= \frac{R^2 - x^2}{R^2 - x^2 - y^2}\,.
\end{aligned}$$

For the parametrization $\boldsymbol{r}(u, v)$, we'll use v as the parameter of the curve. This great circle is the intersection of the hemisphere with the plane $4y = 5z$, which in

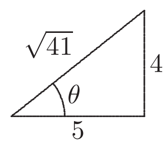

FIGURE 1. Proof that $\arccos(5/\sqrt{41}) = \arctan(4/5)$.

terms of u and v says

$$u = \arctan\left(\frac{4\sin v}{5}\right), \quad du = \frac{20\cos v}{25 + 16\sin^2 v}\, dv\,.$$

When we substitute this value into the expression for G, we find that

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = \frac{1025R^2}{(25 + 16\sin^2 v)^2}\, dv^2\,.$$

For the portion of the curve whose length we want, the parameter v ranges from $\arccos(16/\sqrt{481})$ to $\arccos(-16/\sqrt{481})$, and we find the length to be

$$R\sqrt{1025}\int_{\arccos(16/\sqrt{481})}^{\arccos(-16/\sqrt{481})} \frac{1}{25 + 16\sin^2 v}\, dv = 2R\arctan\left(\frac{16}{3\sqrt{41}}\right).$$

For the parametrization $\boldsymbol{s}(x, y)$, we'll use x as a parameter, so that

$$y = \frac{5\sqrt{R^2 - x^2}}{\sqrt{41}}, \quad dy = -\frac{5s}{\sqrt{41(R^2 - x^2)}}\,.$$

Then we have

$$ds^2 = \frac{R^2}{R^2 - x^2}\, dx^2\,.$$

Since x ranges from $-16R/25$ to $16R/25$, we find the length to be what it should be, namely

$$2R\int_0^{16R/25} \frac{1}{\sqrt{R^2 - x^2}}\, dx = 2R\arctan\left(\frac{16}{3\sqrt{41}}\right).$$

As for the areas, first of all, we know the answer in advance. This area is the segment of the sphere, essentially between two lines of longitude separated by an angle θ whose tangent is $4/5$. The area is therefore $\theta/2\pi$ times the area of the sphere, that is, $2R^2\theta$. To do the computation, we have $\widetilde{E}\widetilde{G} - \widetilde{F}^2 = R^2/(R^2 - x^2 - y^2)$, so that the area computed in the parametrization $\boldsymbol{s}(x, y)$ is said by *Mathematica* to be

$$\int_{-R}^{+R}\int_{\frac{5\sqrt{R^2-x^2}}{\sqrt{41}}}^{\sqrt{R^2-x^2}} \frac{R}{\sqrt{R^2 - x^2 - y^2}}\, dy\, dx = 2R^2\arccos\left(\frac{5}{\sqrt{41}}\right).$$

In the parametrization $\boldsymbol{r}(u, v)$, we have $EG - F^2 = R^4\cos^2 u$, and *Mathematica* gives the area as

$$2R^2\arctan\left(\frac{4}{5}\right).$$

It is easy to verify that the two expressions for the area are the same (see Fig. 1).

PROBLEM 6.5. Verify the relation

$$EG - F^2 = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 (PR - Q^2)\,.$$

Solution: It was established in Problem 6.3 above that

$$\begin{aligned}
E &= P\left(\frac{\partial x}{\partial u}\right)^2 + 2Q\,\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial u} + R\left(\frac{\partial y}{\partial u}\right)2\,,\\
F &= P\,\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v} + Q\left(\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial v} + \frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u}\right) + R\,\frac{\partial y}{\partial u}\,\frac{\partial y}{\partial v}\,,\\
G &= P\left(\frac{\partial x}{\partial v}\right)^2 + 2Q\,\frac{\partial x}{\partial v}\,\frac{\partial y}{\partial v} + R\left(\frac{\partial y}{\partial v}\right)2\,.
\end{aligned}$$

The rest is just some messy algebra, carried out by examining the six terms in the expansion of the product $EG - F^2$ containing the factors P^2, PQ, PR, Q^2, QR, and R^2. In that order, these terms are:

$$P^2\left(\left(\frac{\partial x}{\partial u}\right)^2\left(\frac{\partial x}{\partial v}\right)^2 - \left(\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}\right)\right) = 0\,,$$

$$2PQ\left(\left(\frac{\partial x}{\partial u}\right)^2\frac{\partial x}{\partial v}\,\frac{\partial y}{\partial v} + \frac{\partial x}{\partial u}\left(\frac{\partial x}{\partial v}\right)^2\frac{\partial y}{\partial u} - 2\left(\frac{\partial x}{\partial u}\right)^2\frac{\partial x}{\partial v}\,\frac{\partial y}{\partial v} - 2\frac{\partial x}{\partial u}\left(\frac{\partial x}{\partial v}\right)^2\frac{\partial y}{\partial v}\right) = 0\,,$$

$$PR\left(\left(\frac{\partial x}{\partial u}\right)^2\left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\left(\frac{\partial y}{\partial u}\right)^2 - 2\left(\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u}\,\frac{\partial y}{\partial v}\right)\right.$$

$$= PR\left(\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u}\right)^2,$$

$$Q^2\left(4\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u}\,\frac{\partial y}{\partial v} - \left(\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial v} + \frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u}\right)^2\right)$$

$$= -Q^2\left(\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u}\right)^2,$$

$$2QR\left(\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial u}\left(\frac{\partial y}{\partial v}\right)^2 - \left(\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial v} + \frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u}\right)\frac{\partial y}{\partial u}\,\frac{\partial y}{\partial v}\right) = 0\,,$$

$$R^2\left(\left(\frac{\partial y}{\partial u}\right)^2\left(\frac{\partial y}{\partial v}\right)^2 - \left(\frac{\partial y}{\partial u}\,\frac{\partial y}{\partial v}\right)^2\right) = 0\,.$$

Thus, we get the required relation

$$EF - G^2 = (PR - Q^2)\left(\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u}\right)^2 = (PR - Q^2)\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2.$$

PROBLEM 6.6. Show that a linear combination of tensors of type (k,l) (with addition and scalar multiplication defined componentwise) is a tensor of type (k,l).

Solution: It is routine to show that the sum of two multilinear functions is multilinear, that is, linear in each variable when all the other variables are held fixed. As for the tensor nature of the sum, it again follows from the fact that multiplication by the Jacobian matrix is a linear operation.

PROBLEM 6.7. The *tensor product* of a tensor field S of type (k,l) with components $S^{j_1\dots j_l}_{i_1\dots i_k}$ and a tensor field T of type (p,q) with components $T^{n_1\dots n_q}_{m_1\dots m_p}$ is defined

as the multilinear operator $S \otimes T$ whose effect on a set of $k+p$ vector fields $\boldsymbol{u}_a$ and $l+q$ covector fields $\boldsymbol{v}^b$ is defined by

$$S \otimes T(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{k+m}, \boldsymbol{v}^1, \ldots, \boldsymbol{v}^{l+n}) = \\ = S^{j_1 \ldots j_l}_{i_1 \ldots i_k} T^{n_1 \ldots n_q}_{m_1 \ldots m_p} u_1^{i_1} \ldots u_k^{i_k} u_{k+1}^{m_1} \ldots u_{k+p}^{m_p} v_{j_1}^1 \ldots v_{j_l}^l v_{n_1}^{l+1} \ldots v_{n_q}^{l+q} .$$

Show that $S \otimes T$ is a tensor field of type $(k+p, l+q)$.

Solution: The multilinearity—that is, the linearity in each variable when the others are held fixed—follows immediately from the multilinearity of each of the two factors. The type $(k+p, l+q)$ is obvious from inspection. The tensor nature again follows because each of the factors is a tensor, and multiplication by the Jacobian matrix is a linear operation.

PROBLEM 6.8. If T is a tensor field of type (k, l), where k and l are both positive—let its components be $T^{j_1 \cdots j_l}_{i_1 \ldots i_k}$—the *contraction* of T on a pair of indices i_r, j_s is defined as the multilinear operator whose components are $T^{j_1 \cdots j_{s-1}\, m\, j_{s+1} \cdots j_l}_{i_1 \ldots i_{r-1}\, m\, i_{r+1} \ldots i_k}$, where, of course, summation over m is required. Show that the contraction is a tensor of type $(k-1, l-1)$.

Solution: The multilinearity of the contraction is a trivial consequence of the multilinearity of the original tensor field T and the fact that a sum of multilinear functions is multilinear. As for the tensor nature of the contraction, observe that if T has the components $T^{j_1 \cdots j_l}_{i_1 \ldots i_k}$ in x-coordinates and components $\tilde{T}^{q_1 \cdots q_l}_{p_1 \ldots p_k}$ in y-coordinates, then

$$\tilde{T}^{q_1 \cdots q_l}_{p_1 \ldots p_k} = T^{j_1 \cdots j_l}_{i_1 \ldots i_k} \frac{\partial x^{i_1}}{\partial y^{p_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{p_k}} \frac{\partial y^{q_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{q_l}}{\partial x_{j_l}} ,$$

where the right-hand side, by the Einstein summation convention, is summed over all $i_1, \ldots, i_k, j_1, \ldots, j_l$. When we take $p_r = m = q_s$ and hold these values fixed, the right-hand side contains the product

$$\frac{\partial x^{i_r}}{\partial y^m} \frac{\partial y^m}{\partial x^{j_s}} ,$$

and when this expression is summed over m, the result is

$$\frac{\partial x^{i_r}}{\partial x^{j_s}} = \delta^{j_s}_{i_r} .$$

Thus, the right-hand side is summed only over indices for which $i_r = j_s$, and thus the contraction of the left-hand side is obtained from the contraction of the right-hand side through multiplication of the corresponding Jacobians, the two Jacobians corresponding to the indices of contraction cancelling each other out.

PROBLEM 6.9. Let $G = g_{ij}$ be the metric on a manifold and T a tensor field of type (k, l) where $l > 0$, with the usual components $T^{j_1 \cdots j_l}_{i_1 \ldots i_k}$. As mentioned in the text above, we can "lower" one of the indices j_r by first forming the tensor $G \otimes T$, then contracting it on the lower index j and the upper index j_r. Show that the result is a tensor of type $(k+1, l-1)$. (The covariant Riemann curvature tensor R_{ijkl} is the result of lowering the upper index of the Riemann curvature tensor R^i_{jkl}.)

Solution: The type of the resulting tensor is obvious from inspection. That it is a tensor is easily seen, since it is a contraction of the tensor product $G \otimes T$.

PROBLEM 6.10. Show that for two systems of parameters that are *linearly* related, the Christoffel symbols do transform like a tensor of type $(1, 2)$.

Solution: As shown in Theorem 6.2 of Appendix 6, the term that keeps the Christoffel symbols from transforming like a tensor contains second-order derivatives of the coordinates, and for linear functions the second-order derivatives are all zero.

Subject Index

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